# EXACT RECONSTRUCTION FORMULA FOR DIFFUSE OPTICAL TOMOGRAPHY USING SIMULTANEOUS SPARSE REPRESENTATION

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## ABSTRACT

Diffuse optical tomography (DOT) is a sensitive and relatively low cost imaging modality. However, the inverse problem of reconstructing optical parameters from scattered light measurements is highly nonlinear due to the nonlinear coupling between the optical coefficients and the photon flux in the diffusion equation. Even though nonlinear iterative methods have been commonly used, such iterative processes are computationally expensive especially for the three dimensional imaging scenario with massive number of detector elements. The main contribution of this paper is a novel *non-iterative* and *exact* inversion algorithm when the optical inhomogeneities are *sparsely* distributed. We show that the problem can be converted into simultaneous sparse representation problem with multiple measurement vectors from compressed sensing framework. The *exact* reconstruction formula is obtained using simultaneous orthogonal matching pursuit (S-OMP) and a simple two step approach without ever calculating the diffusion equation. Simulation results also confirm our theory.

***Index Terms***— diffuse optical tomography, compressed sensing, simultaneous sparse representation, s-OMP

## 1. INTRODUCTION

The objective of optical diffusion imaging is to reconstruct the optical properties of cross-section of a highly scattering medium such as tissue, based on measurements of the scattered and attenuated optical energy [1]. Basically, there exists two contrast mechanisms. Intrinsic contrast mechanism relies on the absorption or scattering changes of biological samples due to underlying physiological changes, whereas the extrinsic contrast mechanism uses targeted or activatible probes that emit the fluorescence light. This paper mainly focus on the intrinsic contrast mechanism since it is non-invasive and can be directly used for human cancer or brain imaging without concerning the toxicity of the molecular probes [1].

A major difficulty in optical imaging is, however, that the optical signal within tissue experiences significant scatter. More specifically, the photon path can be approximated using the diffusion equation [1]:

$$(\nabla^2 + \alpha^2(\mathbf{r}))\phi^k(\mathbf{r}) = -S^k(\mathbf{r}) \ , \quad k = 1, \cdots, K \ , \tag{1}$$

where $\phi^k(\mathbf{r})$ denotes the complex modulation envelop of the photon flux at the angular modulation frequency $\omega$ and the position $\mathbf{r} \in \Omega \subset \mathbb{R}^3$. The superscript $k$ indicates the source index in a multi-source imaging scenario. In Eq. (1), $\alpha^2(\mathbf{r}) = (-\mu_a(\mathbf{r}) + j\omega/c)/D(\mathbf{r})$ is the complex wave number with the absorption coefficient $\mu_a(\mathbf{r})$ and the speed of the light in the medium $c$; $S^m(\mathbf{r})$ is the source density, and $D(\mathbf{r})$ is the diffusion constant [1].

Assuming that diffusion constant $D(\mathbf{r})$ is relatively homogeneous, our goal is to estimate the unknown absorption coefficient variation:

$$v(\mathbf{r}) = \begin{cases} \Delta\mu_a(\mathbf{r}) = \mu_a(\mathbf{r}) - \mu_a^0(\mathbf{r}), & \mathbf{r} \in \Theta \\ 0, & \mathbf{r} \in \Omega \backslash \Theta \ . \end{cases} \tag{2}$$

where $\Omega$ denotes the field of view (FOV) and $\Theta$ is a possibly disconnected region of unknown inhomogeneities, respectively. Eq. (1) can then be converted into an integral equation of the second kind [1]:

$$\begin{aligned} \phi^k_{scat}(\mathbf{r}) &= \phi^k(\mathbf{r}) - \phi^k_0(\mathbf{r}) \\ &= \int_\Omega d\mathbf{r}' g_0(\mathbf{r};\mathbf{r}')\phi^k(\mathbf{r})v(\mathbf{r}') \ , \end{aligned} \tag{3}$$

where the homogenous Green's function $g_0(\mathbf{r};\mathbf{r}')$ is calculated by

$$(\nabla^2 + \alpha_0^2(\mathbf{r}))g_0(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{4}$$

and the incident field:

$$\phi^k_0(\mathbf{r}) = \int_\Omega d\mathbf{r}' g_0(\mathbf{r};\mathbf{r}')S^k(\mathbf{r}') \ . \tag{5}$$

The image reconstruction problem is then to estimate unknown scattering coefficient $v(\mathbf{r})$ and the support $\Theta$ from the scattered field measurements $\phi^k_{scat}(\mathbf{r})$.

Note that inverse scattering problem in Eq. (3) is highly nonlinear due to the nonlinear coupling between the unknown coefficient $v(\mathbf{r})$ and the photon flux $\phi^k(\mathbf{r})$ in the diffusion equation. Hence, conventional nonlinear iterative methods requires re-calculating the Greens function during each iteration step, requiring huge computational resources [1]. Recently, non-iterative reconstruction method which need not solve the diffusion equation was proposed based on space-space multiple signal classification (MUSIC) [2]. However, this approach does not guarantee the exact reconstruction of absorption coefficients, and the number of recoverable target is limited by the *minimum* between the source and the detector numbers.

The main contribution of this paper is a novel non-iterative exact reconstruction algorithm using simultaneous sparse representation [2]. Unlike the conventional MUSIC approaches, our sparsity based reconstruction theory guarantees that the maximum number of recoverable targets is up to the *average* of the number of sources and detectors, which is usually much greater than that of MUSIC. Furthermore, the problem can be solved using various simultaneous sparse approximation algorithms such as simultaneous orthogonal matching pursuit (S-OMP) [3], and the unknown scattering coefficients can be then *exactly* calculated based on a simple two step approach without calculating a diffusion equation.

Recently, we proposed a similar non-iterative exact inverse scattering approach for electromagnetic inverse scattering problem [4]. However, the main difference of diffuse tomography from the electromagnetic inverse scattering is that the resultant Green's function is lossy, which makes the measurement basis much more *coherent* with the signal basis. Such coherence results in significant performance degradation of S-OMP. Hence, we proposes a novel normalization and basis transform technique to overcome the issues.

## 2. PROBLEM FORMULATION

With all the measurements from each detectors and sources pairs, we represent an integral nonlinear equation Eq. (3) in a matrix equation:

$$\mathbf{\Phi} = \mathbf{G}\mathbf{X}, \tag{6}$$

where $\mathbf{\Phi} \in \mathbb{C}^{M\times K}$, $\mathbf{G} \in \mathbb{C}^{M\times N}$ and $\mathbf{X} \in \mathbb{C}^{N\times K}$. Here, the elements of each matrix are given as follows: $\Phi_{m,k} = \phi^k_{scat}(\mathbf{r}_m)$, $G_{m,n} = g_0(\mathbf{r}_m;\mathbf{r}_n)$ and $X_{n,k} = \phi^k(\mathbf{r}_n)v(\mathbf{r}_n)$, respectively.

Now we define *row-diversity* measure that counts the number of rows in $\mathbf{X}$ that contain non-zero elements:

$$\mathcal{R}(\mathbf{X}) = \sum_{n=1}^{N} \chi\left[||\mathbf{X}_{n,:}|| > 0\right] \tag{7}$$

where $\chi[\cdot]$ denotes the indicator function and $||\cdot||$ is an arbitrary vector norm. Since $X_{n,k} = \phi^k(\mathbf{r}_n)v(\mathbf{r}_n)$, $||\mathbf{X}_{n,:}|| > 0$ implies that the absorption change $v(\mathbf{r}_n)$ is nonzero. Hence, we identify the row-diversity $\mathcal{R}(\mathbf{X})$ as the number of voxels with non-zero absorption changes. Usually, the absorption coefficients change on a small portion of FOV. For example, in optical breast cancer imaging [5], the main goal is to detect the absorption changes due to the angiogenesis of *sparsely* distributed cancerous cells. Therefore, we can safely assume the sparsity of optical parameter changes.

Under the sparsity condition, the diffuse optical tomography problem can be formulated as following:

$$(\mathbf{P0}): \quad \min\ \mathcal{R}(\mathbf{X}), \quad \text{subject to } \mathbf{\Phi} = \mathbf{G}\mathbf{X}. \tag{8}$$

## 3. NON-ITERATIVE EXACT RECONSTRUCTION

### 3.1. Identifiability

Let $spark(\mathbf{G})$ denote the smallest number of linearly dependent column of $\mathbf{G}$.

**Theorem 3.1** (Chen and Huo [3])**.** *Let $rank(\mathbf{\Phi})$ denotes the rank of the matrix $\mathbf{\Phi}$. Then,* **(P0)** *has the unique solution if*

$$\mathcal{R}(\mathbf{X}) \quad \leq \quad \left(spark(\mathbf{G}) + rank(\mathbf{\Phi}) - \mathbf{1}\right)/2\,. \tag{9}$$

Since $spark(\mathbf{G}) \leq M+1$ and *rank*$(\mathbf{\Phi}) \leq K$, we have the following upper bound:

$$\mathcal{R}(\mathbf{X}) \leq (M+K)/2\ . \tag{10}$$

Note that the maximum number of recoverable absorption spots is up to the *average* of the number of sources and detectors. This is a very big improvement over space space MUSIC in which the maximum number of recoverable targets is bounded by the *minimum* number of detectors or sources.

### 3.2. Coherence of Bases

Suppose $||\mathbf{r}-\mathbf{r}_1|| \ll ||\mathbf{r}-\mathbf{r}_2||$. Since the Green's function of the diffusion equation is diffusive, we have $|g(\mathbf{r};\mathbf{r}_1)| \gg |g(\mathbf{r};\mathbf{r}_2)|$. Therefore, we can easily expect that the magnitude of each column vector of $\mathbf{G}$ varies significantly, hindering the use of greedy algorithm such as S-OMP. To overcome this, the matrix $\mathbf{G}$ is factored as $\mathbf{G} = \mathbf{G}'[\mathbf{W}]$, where each column of $\mathbf{G}'$ is normalized to 1 and the diagonal matrix $\mathbf{W}$ is constructed as $W_{i,i} = ||\mathbf{G}_{:,i}||$, where $\mathbf{G}_{:,i}$ denotes the $i$-th column of the $\mathbf{G}$ matrix. Then, we have the following normalized optimization problem:

$$(\mathbf{P0})': \quad \min\ \mathcal{R}(\mathbf{X}'), \quad \text{subject to } \mathbf{\Phi} = \mathbf{G}'\mathbf{X}', \tag{11}$$

where $\mathbf{X}' = [\mathbf{W}]\mathbf{X}$. However, there still remains another important technical issue. Let *mutual incoherence* of the matrix $\mathbf{G}'$

$$\mu(\mathbf{G}') = \max_{1\leq i,j\leq N, i\neq j} \left|\left\langle \mathbf{G}'_{:,i}, \mathbf{G}'_{:,j}\right\rangle\right|\ , \tag{12}$$

The main technical difficulty in diffuse optical tomography over the electromagnetic scattering problem [4] is that the mutual coherence $\mu(\mathbf{G}')$ is much greater due to the diffusive nature of Green's function. Physically, this is because the detector measurements from adjacent absorption inhomogeneity are measured similarly due to the light scattering.

In order to overcome this technical difficulties, we propose an image basis transform. More specifically, we define a new orthonormal transform matrix $\mathbf{T}$ such that $\mathbf{X}' = \mathbf{T}\mathbf{J}$. Then, the new mutual incoherence can be improved. However, caution should be made in selecting $\mathbf{T}$. Since $\mathbf{X}'$ is sparse, the global transform $\mathbf{T}$ often results in a non-sparse $\mathbf{J}$. Therefore, the resultant imaging problem may not be sparse representation problem. Our goal is, therefore, to select the transform matrix $\mathbf{T}$ that strikes the balance between the sparsity of $\mathbf{J}$ and the resultant mutual incoherence.

### 3.3. Simultaneous Orthogonal Matching Pursuit

Even though **(P0)** gives us the useful information about the uniqueness of the sparse reconstruction, its direct optimization requires computationally expensive combinatorial optimization. For the single measurement vector case such that $N_t = 1$, there have been extensive investigations about so-called $l_0/l_1$-*equivalence* [6]. More specifically, if the measurement basis and the signal basis are *incoherent*, the convex $l_1$ optimization problem provides the unique solution which is identical to that of **(P0)** [6]. Recently, many different approaches has been proposed to extend the $l_1$ minimization idea to address the multiple measurement vector problem [3, 7]. The most popular approaches are based on forward sequential selection methods using simultaneous orthogonal matching pursuit (S-OMP) [3]. For the details of S-OMP, see [3].

### 3.4. Exact Reconstruction of Absorption Coefficients

Recall that our S-OMP algorithm provides us the estimate of active set $\mathcal{I}$ and the corresponding solution $\hat{\mathbf{X}}_{\mathcal{I}}$:

$$\hat{\mathbf{X}}_{\mathcal{I}} \quad = \quad \mathbf{G}_{\mathcal{I}}^{\dagger}\mathbf{\Phi} \tag{13}$$

where the active index is defined as

$$\mathcal{I} = \{n \in \{1, 2, \cdots, N\} : ||\mathbf{X}_{n,:}|| > 0\} \;, \tag{14}$$

However, our imaging goal is to find the absorption coefficient distribution $v(\mathbf{r})$ rather than the induced current distribution $X^k(\mathbf{r}) = \phi^k(\mathbf{r})v(\mathbf{r})$. Since $\phi^k(\mathbf{r})$ is the unknown total flux, in order to calculate $v(\mathbf{r})$ from $X^k(\mathbf{r})$, we need to estimate the total flux $\phi^k(\mathbf{r})$. Interstingly, this can be solved relatively easily using the following two-step approach. The integral equation Eq. (3) tells us that the estimate of the total flux is given by

$$\hat{\phi}^k(\mathbf{r}) \quad = \quad \phi_0^k(\mathbf{r}) + \int_{\Omega} d\mathbf{r}' g_0(\mathbf{r};\mathbf{r}')\hat{X}^k(\mathbf{r}') \;, \tag{15}$$

where $\phi_0^k$, $g_0$, and $\hat{X}^k(\mathbf{r}')$ are already known values. Now, using the relation $X^k(\mathbf{r}) = \phi^k(\mathbf{r})v(\mathbf{r})$, $\Delta\mu_a$ can be readily calculated using the least square fitting:

$$\Delta\hat{\mu}_a(\mathbf{r}) = \frac{\sum_{k=1}^{K} \left(\hat{\phi}^k(\mathbf{r})\right)^* \hat{X}^k(\mathbf{r})}{\sum_{k=1}^{K} |\hat{\phi}^k(\mathbf{r})|^2}, \quad \mathbf{r} \in \Theta. \tag{16}$$

With increasing number of sources (i.e. as $K$ grows), the final estimate of $\Delta\mu_a$ becomes more accurate.

## 4. NUMERICAL RESULTS

In order to validate our findings, we have conducted several experiments. First, our reconstruction algorithm has been used to recover randomly distributed absorption spots in a three dimensional space. The field of view (FOV) for the simulation is $8cm \times 8cm \times 8cm$, which is discretized into $5mm \times 5mm \times 5mm$ voxels. On each face of cube, 256 detectors are uniformly distributed, resulting total 1536 detectors. The modulation frequency of the source was 100 MHz. Figs. 1(a)-(d) illustrate randomly distributed absorption spots, and S-OMP reconstruction results using one, two, and six sources, respectively. We can observe that more source improves the reconstruction quality in general. In this particular example, with six sources located at the center of faces of the cube, the perfect reconstruction was possible.

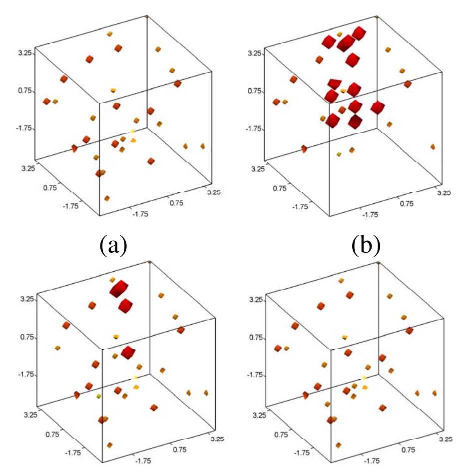


**Fig. 1**. (a) Ground-truth absorption phantom, and S-OMP reconstruction using (b)one, (c)two, (d) and six sources, respectively.

Fig. 2 illustrates the perfect reconstruction ratio of S-OMP with respect to the number of sources and absorption spots. As the number of sources grows, perfect reconstruction ratio increases.

Another simulation was conducted to recover the clustered absorption inhomogeneities. This situation is more common in practise; however, such clustering tends to make the measurement basis coherent with the image basis; hence, the S-OMP performance may be significantly degraded. In order to overcome this problem, Haar wavelet basis are used for image basis, and the S-OMP is applied to reconstruct the wavelet coefficients. The FOV is the same as the previous simulation, and a plane shaped absorption inhomogeneity is located at the center of the FOV as shown in Fig. 3(a). The support size of the object is 178 voxels. Under this simulation condition, the MUSIC approach [2] fails as shown in Fig. 3(d). However, S-OMP reconstruction in Fig. 3(b)-(c) show that the performance is improved with increasing number of sources. The reconstruction results clearly demonstrate the shape of wings and tail.

Note that in this simulation the number of sources $K = 6$, and the number of detectors $M = 1536$; hence the maximum number of reconstructible absorption inhomogeneity is $(6 + 1536)/2 = 771$ voxels. However, the simulation results still exhibits a gap from the upper-bound. The main reason for that is that the upper-bound is for the $l_0$ minimization, and the bound for the S-OMP should be much smaller. Furthermore, the coherence between the measurement and signal bases may contribute the performance loss.

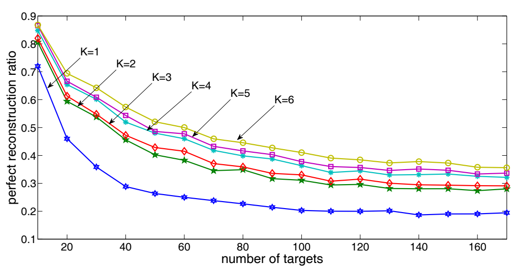


**Fig. 2**. Perfect reconstruction ratio of S-OMP.

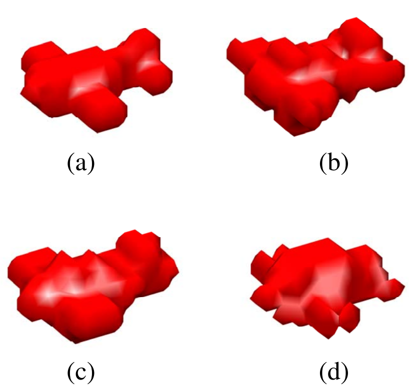


**Fig. 3**. (a) Ground-truth absorption phantom, and S-OMP reconstruction using (b)six, and (c) ninety six sources, respectively. (d) Space space MUSIC reconstruction.

## 5. CONCLUSION

This paper described a novel non-iterative exact reconstruction formula for diffuse optical tomography. By exploiting the sparsity of the absorption changes in practical applications, we formulated the reconstruction problem in the form of the simultaneous sparse representation problem with multiple measurement vectors. S-OMP algorithm was first used to reconstruct the sparsely distributed induced current; then, two step approach was implemented to reconstruct the unknown absorption coefficients. Under the noiseless measurement scenario, exact reconstruction of absorption coefficients was guaranteed without calculating the diffusion equation. The algorithm is ideally suited for three dimensional reconstruction from massive detector elements such as CCD camera, since the total recoverable absorption target is up to the average number of source and detector elements; and we do not need to calculate the diffusion equation. Numerical results also confirmed out findings.

## 6. REFERENCES


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