Noise Variance Analysis of an Optimal Spatio-Temporal Encoding Scheme for Dynamic MRI using Phase Array Coils

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Abstract— **For high quality MR imaging of time-varying objects such as beating heart or brain hemodynamics, fast signal acquisition is required without sacrificing the spatial resolution. Recently, we proposed a novel parallel imaging algorithm for dynamically varying objects. The new algorithm, called x-f SENSE (x-f space SENSitivity Encoding), optimally reduces the data acquisition rate by exploiting the temporal redundancies and sensitivity diversities in a phase array coil. The acceleration factor of the x-f SENSE is up to the product of those of the parallel imaging algorithm and the optimal spatio-temporal encoding algorithm. Furthermore, the reconstruction algorithm of x-f SENSE is as simple as the ordinary SENSE. Due to its simplicity and the optimal acceleration factor, x-f SENSE can be immediately used in practice. However, reconstruction image quality is very dependent on the measurement noise level. This paper is devoted to evaluate the reconstruction noise variance of the x-f SENSE by analyzing the coil sensitivity and noise covariance between the coils. Based on the performance in noisy environments, we again confirm that the x-f SENSE is a very powerful algorithm for fast varying dynamic images in practice.**

Keywords— **Parallel Imaging, SENSE, MRI, Dynamic imaging**

I. INTRODUCTION

Dynamic MRI involves the acquisition and reconstruction of time varying images such as beating hearts or brain hemodynamics. Fast imaging sequence such as echo-planar imaging (EPI)[1] has been widely used in practise; however, EPI sacrifices the image quality to achieve high temporal resolution. Instead of solely resorting to fast imaging sequence, considerable research efforts have been made to produce high temporal and spatial resolution images by exploiting inherent redundancies during data acquisition. For example, parallel imaging methods such as SMASH (SiMultaneous Acquisition of Spatial Harmonics)[2], SENSE (SENSitivity Encoding) [3], and GRAPPA (GeneRalized Autocalibrating Partially Parallel Acquisitions)[4] reduce the scan time by skipping the phase encoding lines in the k-space by exploiting the diversity of coil sensitivity information in phase array RF coils. Unlike the parallel imaging method, temporal filtering techniques such as UNFOLD (UNaliasing by Fourier-encoding the Overlaps Using the temporal Dimension) [5] exploit the temporal redundancies of time varying objects. Many research groups have tried to combine the parallel imaging methods with temporal filtering techniques. Recently, k-t BLAST/SENSE [6] has been proposed, which demonstrated improved reconstruction performance of dynamic imaging from highly undersampled signal acquisition. The main idea of k-t SENSE is to use training data set to find the weighting factor which can be used during the signal reconstruction phase.

Recently, two groups independently proposed the optimal dynamic imaging algorithm using parallel RF coils based on lattice sampling theory [7,8]. More specifically, the optimal imaging problem is formulated as an optimal lattice sampling pattern design problem in k-t space such that the number of spectrum overlap in x-f space is less than the number of coils. Then, the aliasing component can be unfolded by simple matrix inversion of low dimensional matrices, whose computational burden is negligible. Furthermore, the acceleration factor of the two approaches can be up to the product of the number of coils and that of the optimal time-sequential sampling, which is greater than that of k-t SENSE.

In practice, the reconstruction image quality is very much dependent on the noise level. To evaluate reconstruction image quality in advance, SENSE uses “g-factor”, which quantifies the noise amplification. The main goal of this paper is to develop an expression of the reconstruction noise variance to quantify the stability of the x-f SENSE algorithm from noisy measurement.

II. THEORY

A. x-f SENSE

Let $i(t,x)$ denote the time varying proton density image at a time instant t and a spatial location x. In parallel MRI, reconstructed image is usually given by $i(t,x)$ multiplied by the coil sensitivity pattern $\alpha(t,x)$ [3]. Following the usual assumption that the coil sensitivity $\alpha(x,t)$ is time invariant [3], we can represent the measured signal $s(k,t)$ as a 2-D inverse Fourier transform

$$s(k,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(x)I(x,f)e^{-j2\pi(kx+tf)}dxdf, \quad (1)$$

where $I(x,f)$ denotes Fourier transform of the proton density $i(x,t)$ with respect to time. Suppose the full resolution MR image is obtained from samples on the densest lattice Γ. Our goal is to design a sublattice sampling pattern Λ such that the number of aliasing in x-f domain is not bigger than the number of coils such that the aliasing can be unfolded using coil sensitivity.

With a slight abuse of notation, let $\mathbf{t}=(k,t)$ denote the k-space and temporal coordinate and $\mathbf{f}=(x,f)$ denote the coordinate in its reciprocal space, respectively. Then, $\{s_k(\mathbf{t}),\mathbf{t}\in\Lambda\}$ denotes the data samples on a sublattice Λ measured by the k-th coil, where $k=1,\cdots,K$. If we define $Y_k(\mathbf{f})$ as its Fourier transform, we have [9]:

$$Y_k(\mathbf{f})=\sum_{\mathbf{s}\in\Gamma^*}\hat{Y}_k(\mathbf{f}-\mathbf{s}),\quad \mathbf{f}\in U \quad (2)$$

where U denotes the *unit cell* on the reciprocal lattice Γ^* and

$$\hat{Y}_k(\mathbf{f})=\frac{d(\Gamma)}{d(\Lambda)}\sum_{i=1}^{P}\alpha_k(\mathbf{f}-\mathbf{r}_i)I(\mathbf{f}-\mathbf{r}_i), \quad (3)$$

and $P=\frac{d(\Lambda)}{d(\Gamma)}$ corresponds to the acceleration factor, and $\mathbf{r}_i\in\Lambda^*, i=1,\cdots,P$. Now, define the *coil sensitivity pattern* K and the *spectral replica pattern*:

$$K=\{\alpha_1,\cdots,\alpha_K\}, R=\{\mathbf{r}_1,\cdots,\mathbf{r}_P\}. \quad (4)$$

Note that the acceleration factor P is now equal to the number of overlap due to aliasing, i.e. $P=|R|$. Then, Eq. (3) can be represented as

$$\mathbf{Y}(\mathbf{f};K)=\frac{1}{P}\mathbf{W}(K,R)\,\mathbf{I}(\mathbf{f};R) \quad (5)$$

where

$$\mathbf{Y}(\mathbf{f};K)=[\hat{Y}_1(\mathbf{f}),\cdots,\hat{Y}_K(\mathbf{f})]^T \quad (6)$$

$$\mathbf{I}(\mathbf{f};R)=[I(\mathbf{f}+\mathbf{r}_1),\cdots,I(\mathbf{f}+\mathbf{r}_P)]^T \quad (7)$$

and the *aliasing structure matrix* $\mathbf{W}(K,R)$ is given by

$$\mathbf{W}(K,R)=\begin{bmatrix}\alpha_1(\mathbf{f}+\mathbf{r}_1) & \alpha_1(\mathbf{f}+\mathbf{r}_2) & \cdots & \alpha_1(\mathbf{f}+\mathbf{r}_P)\\ \alpha_2(\mathbf{f}+\mathbf{r}_1) & \alpha_2(\mathbf{f}+\mathbf{r}_2) & \cdots & \alpha_2(\mathbf{f}+\mathbf{r}_P)\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_K(\mathbf{f}+\mathbf{r}_1) & \alpha_K(\mathbf{f}+\mathbf{r}_2) & \cdots & \alpha_K(\mathbf{f}+\mathbf{r}_P)\end{bmatrix}$$

If the number of coil and the number of spectral repetition are identical (i.e. $K=P$), Eq. (5) has the unique solution. This corresponds to the conventional parallel imaging and the maximum acceleration factor is equal to the number of coils. However, if the spatio-temporal support is sparse (for example, in the cardiac imaging or fMRI cases where the temporal varying parts are small), further acceleration is possible. We showed that by properly choosing sampling lattice the multifold reduction of data acquisition is possible using x-f SENSE and such reduction factor can achieve the theoretical maximum [7]. This is summarized as following:

Theorem 2.1. [7] *Suppose* K *is defined by Eq. (4) and* M *is given by*

$$M=\{\mathbf{r}_i\in\Lambda^*: I(\mathbf{f}+\mathbf{r}_i)\ \text{is non-zero}\} \quad (8)$$

Suppose, furthermore, $K=|K|$ *and* $M=|M|$*. If coil sensitivity pattern* K *is* (K,M) *universal and* $M=K$ *for all* $\mathbf{f}$*, x-f SENSE is perfectly reconstructible and achieves the optimal acceleration factor*.

Amazingly, the universality of the matrix is quite general and with overwhelming probability the randomly selected matrix satisfies the universality condition [10]. Hence, we conjecture that randomly selected phase array coil sensitivity pattern usually satisfies the universality condition with high probability. Our simulation result with the sensitivity map from a real scanner also confirms the conjecture. The x-f SENSE reconstruction algorithm is, then, similar to the conventional SENSE algorithm. Theorem 2.2 explicitly shows how much acceleration x-f SENSE can achieve for a given x-f support.

Theorem 2.2. *Suppose* S *denotes the non-zero support of the image in x-f domain. The maximal acceleration factor of x-f SENSE is then given by*

$$P\le K\frac{\lambda(FOV)}{\lambda(S)} \quad (9)$$

where $\lambda(\cdot)$ *denotes the Lebesque measure, FOV denotes the field of view in x-f domain and* K *denotes the number of RF coils, respectively.*

The factor $\lambda(FOV)/\lambda(S)$ is called the *Landau rate* in the non-uniform sampling theory, which is the maximum acceleration factor that the optimized spatio-temporal sampling [11,12] can achieve. Therefore, Theorem 2.2 reveals that the maximal acceleration factor of x-f SENSE is up to the product of the maximal acceleration factor of the optimized spatio-temporal sampling and that of parallel imaging. Hence, we clearly see that x-f SENSE achieves the acceleration factor that UNFOLD or SENSE alone cannot achieve.

B. Time Sequential Sampling Constraint

The k-t space data acquisition procedure is completely defined by the time-sequential (TS) sampling schedule: $\Psi\{(k_l,t_l): t_{l+1} > t_l\}_{l=-\infty}^{\infty}$ which specifies the k_y phase encoding line and the phase interval, respectively, allowing acquisition of one phase encoding line at each time instant. Willis and Bresler [11 showed that the TS schedule can be satisfied by imposing a constraint on a sampling matrix $\mathbf{A}$. Using these concepts, Aggarwal, Zhao and Bresler [12] showed that the sampling matrix $\mathbf{A}$ should have the following form:

$$\mathbf{A} = \begin{pmatrix} 2k_{ymax} & a \\ 0 & T_R \end{pmatrix} \tag{10}$$

where $2k_{ymax}$ denotes the distance in k-space in the y-direction and T_R denotes the repetition time, respectively. Hence, we use this to optimize the x-f SENSE sampling pattern.

C. Noise Variance Image

Until now, we have been mainly concerned about the perfect reconstruction condition from noise free measurements. However, real life measurements are always noisy, and we are more concerned about the effects of measurement noise in the reconstructed images. This is the main reason why so-called "g-factor" analysis is important in parallel imaging [3].

This section is devoted to analyze the reconstruction noise variance of our new x-f SENSE algorithm. Let $\Psi \in C^{K\times K}$ denotes the noise covariance matrix between K number of RF coils. Here, we assume that the noise covariance matrix is independent on image pixel location. Then, Theorem 2.3 explains how the noise variance map of x-f SENSE can be calculated in x-t space.

Theorem 2.3. *Suppose* $(x_i,t_j), i=1,\cdots,N_X, j=1,\cdots,N_T$ *denote the image sample coordinates in x-t space, where* N_T *and* N_X *denote the number of time frames and phase encoding step, respectively. Then, the x-f SENSE reconstruction noise variance* $\sigma^2(x_i,t_j)$ *in x-t space is given by*

$$\sigma^2(x_i,t_j) = \frac{1}{N_T}\sum_{k=1}^{N_T}\sigma^2(x_i,f_k) \tag{11}$$

where $\sigma^2(x_i,f_k)$ *is given by*

$$\begin{cases} 0, & \text{if no signal at } (x_i,f_k) \\ P^2\left(\left(\mathbf{S}^{i,k}\right)^H \Psi^{-1}\left(\mathbf{S}^{i,k}\right)\right)^{-1}_{(1,1)}, & \text{otherwise.} \end{cases}$$

where $\mathbf{S}^{i,k}$ *denotes the active sensitivity matrix at the location* (x_i,f_k) *and the subscript* $(1,1)$ *denotes the (1,1) elements of the matrix.*

Note that Eq. (11) averages out the noise variance in x-f domain along f direction. This minimizes the noise fluctuation along temporal direction, hence noise pattern of x-f SENSE is less noticeable along time.

D. Numerical Results

Numerical simulations are conducted to demonstrate the performance of the x-f SENSE. In this example, the phantom is again composed of three static and one time varying circles. The magnitude of the time varying component changes between $-5.4s$ to $7.39s$, and the maximum size x-f support is given in Fig. 1. As a ground truth time varying image, the original phantom images are illustrated in Fig. 2. We again use two coils with the sensitivity map as shown in Fig. 3.

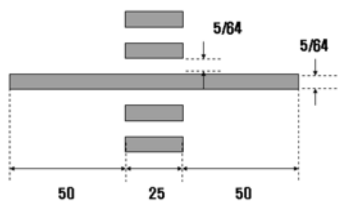

Figure 1. The x-f support of simulation phantom.

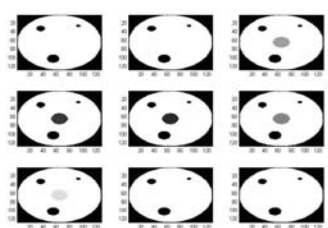

Figure 2. The time series of original phantom.

The optimal sampling matrix $\mathbf{A}$ in Eq. (10) is designed in the x-f space such that it allows up to two spectral overlaps. Then, the resultant sampling matrix in k-t space, we have

$$\mathbf{A} = \begin{pmatrix} 2k_{ymax} & a \\ 0 & T_R \end{pmatrix} = \begin{pmatrix} 1 & \frac{5}{128} \\ 0 & \frac{1}{10} \end{pmatrix} \tag{12}$$

which implies that $k_{ymax} = 0.5mm^{-1}$ and $T_R = 0.1s$. Therefore, the resultant sampling schedule is given by

$$\{(k_l, t_l)\}_{l=-\infty}^{\infty} = \left\{ \left(\frac{(5l \bmod 128)}{128}, \frac{1}{10} l \right) \right\}_{l=-\infty}^{\infty}. \tag{13}$$

The reconstructed images are then artifact free and identical to Fig. 2.

For comparison, the optimized time sequential sampling pattern with a single coil can be found by computing the reciprocal lattice without aliasing and converting it to a sampling matrix. The resultant optimal sampling matrix for the given support with 128 phase encoding step is given by

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{k}{128} \\ 0 & \frac{1}{30} \end{pmatrix} \tag{14}$$

where $k \le 128$ is any odd number. Hence, we can find that the optimal $T_R = \frac{1}{30}s$, which corresponds to three times faster sampling schedule compared to that of x-f SENSE. Hence, we can conclude that our x-f SENSE significantly relaxes the sampling rate constraint.

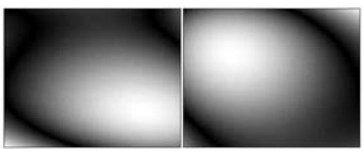

Figure 3. The sensitivity maps of coil 1 and coil 2.

Fig. 4 illustrates the reconstruction noise variance image of x-f SENSE calculated based on Theorem 2.3. The white half circular area in the static background is noticeable due to low sensitivity of the two coils as can be shown in Fig. 3. Another noticeable difference in this variance image is that it shows “ghost” images in the static background. The artifacts could be traced back to potential aliasing artifact in the x-f domain in case of noisy measurements.

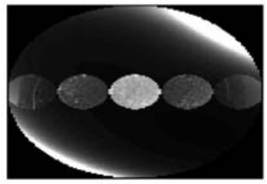

Figure 4. Noise variance map for the reconstruction.

III. Conclusions

This paper proposes a novel parallel imaging algorithm, called x-f SENSE, that optimally combines the parallel imaging and the optimized time sequential sampling using the powerful lattice sampling theory. Unlike the conventional algorithms, x-f SENSE achieves the unprecedent acceleration in data acquisition by exploiting the temporal redundancy of data and the diversity of the coil sensitivity. More specifically, the maximal acceleration factor of the x-f SENSE is given by the product of the maximal acceleration factor of the optimized spatio-temporal sampling alone and that of parallel imaging. Furthermore, the reconstruction procedure of x-f SENSE is as simple as that of the popular SENSE algorithm, so it can be immediately used in practise. Noise variance analysis has been conducted, which showed that the noise pattern in x-f SENSE is smoothed along temporal axis, resulting improved visual quality. Numerical simulation demonstrated that x-f SENSE is very powerful parallel imaging method for dynamic MRI.

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