

SINGLE CHANNEL EXACT 3-D BLIND IMAGE DECONVOLUTION FROM CYLINDRICALLY SYMMETRIC BLUR KERNEL

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ABSTRACT

The rotational symmetry of point spread function (PSF) is common for many imaging systems such as optical microscopes, cameras, astigmatism corrected electron microscopes, and etc. In 2-D deconvolution problem, we showed that an exact PSF estimation from a single measured image on a flat background is possible by exploiting the circular symmetry. This paper extends the idea to 3-D blind deconvolution problem. More specifically, we show that the exact recovery of 3-D PSF is possible from a single set of z-stack images if the PSF of the microscope has a cylindrical symmetry and there exists enough working distance to cover the volumetric sample from top to bottom. The resulting algorithm allows an accurate computational optical sectioning of biological specimen from z-stack images using brightfield or fluorescence microscopes without separate PSF measurements. Experimental results confirm our theory.

Index Terms— 3-D blind deconvolution, subspace method, block-Toeplitz matrix

1. INTRODUCTION

Blind image deconvolution is termed for identification and correction of a point spread function (PSF) of an image acquisition device based on measured data under limited prior knowledge of PSF. Even though this problem may be categorized as a classical one, novel solutions have been continuously proposed. In this paper, we show that the cylindrical symmetry of 3-D point spread function (PSF) in optical microscopy is an important clue for accurate blind deconvolution.

As described in our previous 2-D blind deconvolution works [1], this idea was inspired by *multichannel* blind image deconvolution which allows the exact recovery of unknown blur kernels when multiple measurements of an identical scene through distinct blur channels are available [2]. However, in many experimental situations, it is difficult to obtain multiple distinct blur measurements of an identical scene.

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For example, in fluorescence microscopy, the fluorescence intensity could vary due to fast physiological changes and photobleaching between the multiple acquisitions.

For 2-D blind deconvolution problem, we showed that the exact recovery of PSF is still possible using a non-iterative algorithm from *single* channel measurement when the PSF has radial symmetry and the support region of measurement is surrounded by a flat background [1]. By Fourier-slice theorem, the single-input single-output (SISO) problem in spatial domain can be converted into a multiple-input multiple-output (MIMO) problem in Radon domain. The unknown blur kernel is then estimated from projection data at distinct views by the subspace method without any iterative step [3]. However, the theory fails when we can only measure a part of true image, which is common in practice [2].

In this paper, we extend the 2-D theory to 3-D deconvolution algorithm by exploiting the cylindrical symmetry of the 3-D PSF function. Unlike the 2-D cases, the unknown PSF can be shown to be *exactly* estimated even from partial data measurements as long as there is no information loss in axial direction, which is the case when the working distance of microscope is long enough to cover the 3-D sample from top to bottom. The exactness of estimated solution by subspace method is due to the block Toeplitz structure of 2-D convolution matrix, as will be shown in this paper. Since a PSF of microscopy is usually cylindrically symmetric and 3-D deconvolution is an important tool for computational optical sectioning, our theory may have significant impact in many real applications.

2. NOTATION

Two letters M and N are reserved for expressing the number of rows and columns of a matrix, respectively. The letter K is reserved for the number of projection angles in Radon transform. The letter $\mathbf{H} \in \mathbb{R}^{m \times n}$ stands for a matrix that contains 2-D shift invariant PSF. The row direction filters are $\{\mathbf{h}_i\}_{i=1}^n$, and the column directional filters are denoted by $\{\mathbf{H}_j\}_{j=1}^m$.

Similar to the notation by Harikumar and Bresler [2], we define 1-D convolution matrices. $\mathbf{C}_{M_x}\{\mathbf{h}_i\}$ denotes 1-D full convolution matrix of which blur kernel is $\mathbf{h}_i \in \mathbb{R}^m$ and the

number of row and column vectors is $M_O = M_I + m - 1$ and M_I , respectively. More specifically, for a given input signal length M_I , a full 1-D convolution matrix $\mathbf{C}_{M_I}\{\mathbf{h}_i\}$ is defined as

$$\mathbf{C}_{M_I}\{\mathbf{h}_i\} = \begin{bmatrix} \mathbf{h}_i(1) & & & & 0 \\ \vdots & \ddots & & & \\ \mathbf{h}_i(m) & & \ddots & & \\ & & & \ddots & \mathbf{h}_i(1) \\ & & & & \vdots \\ 0 & & & & \mathbf{h}_i(m) \end{bmatrix} \quad (1)$$

where $\mathbf{h}_i(k)$ denotes the k -th elements of \mathbf{h}_i .

We define 2-D convolution matrices in similar ways. Without loss of generality, there exist three categories of 2-D convolution matrices. First, full data acquisition in both row and column directions; second, full in row and chopped in column direction, respectively; finally, chopped in both directions. The case in which the row direction is chopped but full in column direction can be converted to the second case by transposing the image. Now, the convolution matrix for our interest is the second case given by

$$\begin{bmatrix} \mathbf{C}_{M_I}\{\mathbf{h}_n\} & \cdots & \mathbf{C}_{M_I}\{\mathbf{h}_1\} & & & & 0 \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ 0 & & & & & \mathbf{C}_{M_I}\{\mathbf{h}_n\} & \cdots & \mathbf{C}_{M_I}\{\mathbf{h}_1\} \end{bmatrix} \quad (2)$$

where the output image dimensions are given by $M_O = M_I + n - 1$ and $N_O = N_I - n + 1$.

3. PROBLEM FORMULATION

In most of the computationally optical sectioning techniques, the PSF is considered shift-invariant and cylindrical symmetric. More specifically, let $\mathbf{r} \in \mathbb{R}^3$ denote the 3-D Cartesian coordinate. Then, for a true volume $x(\cdot)$, blurred measurement $y(\cdot)$ filtered by a 3-D PSF $h^{2D}(\cdot)$ is then described by a 3-D convolution:

$$y(\mathbf{r}) = (h^{3D} *** x)(\mathbf{r}) + n(\mathbf{r}), \quad (3)$$

where $***$ denotes the 3-D convolution operation and $n(\cdot)$ is the additive noise. This can be represented in Fourier domain as follows:

$$Y(k_x, k_y, k_z) = H^{3D}(k_x, k_y, k_z)X(k_x, k_y, k_z) + N(k_x, k_y, k_z), \quad (4)$$

where (k_x, k_y, k_z) denote 3-D Cartesian coordinate in Fourier domain, $Y(\cdot)$, $H^{3D}(\cdot)$, $X(\cdot)$, and $N(\cdot)$ denotes the 3-D Fourier transform of measurement, PSF, true z-stack, and noise, respectively. We can derive an equivalent equation on the cylindrical coordinate:

$$Y(k_r, \Theta, k_z) = H^{3D}(k_r, \Theta, k_z)X(k_r, \Theta, k_z) + N(k_r, \Theta, k_z), \quad (5)$$

where $k_r = (k_x^2 + k_y^2)^{\frac{1}{2}}$ and $\Theta = \tan^{-1} \frac{k_y}{k_x}$, respectively. By employing the cylindrical symmetry of a PSF, i.e., $H^{3D}(k_r, \Theta, k_z) = H^{2D}(k_r, k_z)$, the 3-D image convolution is then reduced to a separable 2-D convolution at each Θ :

$$y^\Theta(r, z) = (h^{2D} ** x^\Theta)(r, z) + n^\Theta(r, z), \quad (6)$$

where $**$ denotes the 2-D convolution operation, $y^\Theta(\cdot)$, $x^\Theta(\cdot)$, and $n^\Theta(\cdot)$ denotes inverse Fourier transform pair of $Y(\cdot)$, $X(\cdot)$, and $N(\cdot)$ for a given angle Θ , respectively. Therefore, similar to 2-D blind deconvolution with circular symmetry [1], the 3-D SISO problem in spatial domain can be translated into 2-D MIMO problem through a single blur kernel $h^{2D}(\cdot)$ in (r, z) -plane when we consider each $y^\Theta(r, z)$ as a 2-D measurement.

Especially important acquisition scheme is the case where the axial data along z -direction is full whereas the measurement is partial along lateral direction. More specifically, in optical microscopy with sufficient working distance, we can obtain the additional out-of-focus z -slice images above and below the actual physical supports of the specimen. This additional acquisition makes the measurements full data along z -direction even though the measurements is still partial along r -direction. More specifically, in discrete (r, z) -plane, the measured plane $\mathbf{Y}^{(i)}$ corresponding to the i -th projection angle can be described as the 2-D chopped convolution between the unknown blur kernel $\mathbf{H} = [\mathbf{h}_1, \dots, \mathbf{h}_n] \in \mathbb{R}^{m \times n}$ and the true image $\mathbf{X}^{(i)} = [\mathbf{x}_1, \dots, \mathbf{x}_{N_I}] \in \mathbb{R}^{M_I \times N_I}$ as follows:

$$\text{VEC}\{\mathbf{Y}^{(i)}\} = \mathfrak{C}_{(M_I, N_I)}^{f,c}\{\mathbf{H}\} \cdot \text{VEC}\{\mathbf{X}^{(i)}\} \quad (7)$$

4. MAIN RESULTS

In this section, we introduce a sequence of results that leads to the exact recovery condition for 3-D deconvolution problem from partial data. Due to the page limitation, all the proof is skipped in this paper.

Lemma 1 *The 2-D convolution matrix $\mathfrak{C}_{(M_I, N_I)}^{f,c}\{\mathbf{H}\}$ for partial data case has full column rank for generic $\mathbf{H} \in \mathbb{R}^{m \times n}$ if*

$$\frac{N_I}{M_I} \geq \frac{n-1}{m-1}. \quad (8)$$

Physically, the necessary condition $N_I/M_I \geq (n-1)/(m-1)$ for full rankness is very mild and easy to meet in optical microscopy experiments. More specifically, the specimen dimension along lateral direction is usually significantly larger than that of axial one. Therefore, nearly every acquisition scheme in optical microscopy can satisfy the necessary condition $N_I/M_I \geq (n-1)/(m-1)$.

For full data case both in axial and lateral directions, the cross relationship does hold between each (r, z) projections and measurements, similar to multichannel blind deconvolution problem in Harikumar and Bresler [2]. However, the

unknowns in partial data problem are the projections images whose dimension is bigger than the measurements; hence, the cross relationship does not hold in our problem setup.

Therefore, the direct estimation of projection images for the partial data case is hopeless; hence, we are interested in estimating the filter first, and then estimate the images using non-blind deconvolution techniques. We first need to exploit the following properties of the block-Toeplitz matrix.

Lemma 2 Let $\mathbf{Y} (1 : M_{\mathcal{O}}, i : j)$ denote the $M_{\mathcal{O}} \times (j - i + 1)$ submatrix of \mathbf{Y} . Then, we have

$$\begin{aligned} & \text{VEC}\{\mathbf{Y} (1 : M_{\mathcal{O}}, i : j)\} \\ &= \mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\} \cdot \text{VEC}\{\mathbf{X} (1 : M_{\mathcal{I}}, i : j + n - 1)\}, \end{aligned} \quad (9)$$

where $M_{\mathcal{I}} = M_{\mathcal{O}} - m + 1$ and $S = j - i + 1$, respectively.

Thanks to Lemma 2, we have the following relationship

$$\begin{aligned} \mathbf{Y}_{\mathfrak{S}}^Y &= \left[\mathfrak{S}_S\{\mathbf{Y}^{(1)}\}, \dots, \mathfrak{S}_S\{\mathbf{Y}^{(K)}\} \right] \\ &= \mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\} \cdot \left[\mathfrak{S}_S\{\mathbf{X}^{(1)}\}, \dots, \mathfrak{S}_S\{\mathbf{X}^{(K)}\} \right] \\ &\triangleq \mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\} \cdot \mathbf{Y}_{\mathfrak{S}}^X \end{aligned} \quad (10)$$

where the matrix $\mathbf{Y}_{\mathfrak{S}}^Y \in \mathbb{R}^{M_{\mathcal{O}}S \times (N_{\mathcal{O}} - S + 1)K}$ and $\mathbf{Y}_{\mathfrak{S}}^X \in \mathbb{R}^{M_{\mathcal{I}}S \times (N_{\mathcal{I}} - S + 1)K}$ denotes the measurement and unknown projection matrices, respectively; and $\mathfrak{S}_S\{\mathbf{Y}^{(i)}\}$ which constructs a collection of vectorized submatrices $\mathbf{Y}^{(i)} (1 : M_{\mathcal{O}}, i : i + S - 1), i = 1, \dots, N_{\mathcal{O}} - S + 1$.

Lemma 3 $\mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\}$ and $\mathfrak{C}_{(M_{\mathcal{I}}, S+n-2)}^{f,c} \{\mathbf{H}\}$ have full column rank for generic \mathbf{H} as long as $S \geq \frac{M_{\mathcal{I}} - m + 1}{m - 1} (n - 1) + 1$.

Now, we need to check whether a set of basis vectors that construct the subspace of $\text{COL}\{\mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\}\}$ are known. Lemma 4 provides the uniqueness of the column space.

Lemma 4 Let $N_{\mathcal{I}}$ denote any column dimension of the input image such that the convolution matrix $\mathcal{H}_c \triangleq \mathfrak{C}_{(M_{\mathcal{I}}, N_{\mathcal{I}})}^{f,c} \{\mathbf{H}\}$ and $\mathfrak{C}_{(M_{\mathcal{I}}, N_{\mathcal{I}}-1)}^{f,c} \{\mathbf{H}\}$ have full rank. For another blur kernel \mathbf{H}' with the same dimension as \mathbf{H} , let the corresponding convolution matrix be defined as $\mathcal{H}'_c \triangleq \mathfrak{C}_{(M_{\mathcal{I}}, N_{\mathcal{I}})}^{f,c} \{\mathbf{H}'\}$. Then, \mathcal{H}'_c shares the same column space with \mathcal{H}_c if and only if the corresponding blur kernels are proportional, i.e., $\mathbf{H}' = \alpha \mathbf{H}$ for some scalar $\alpha \in \mathbb{R}$.

We now define a simplified notation:

$$\bar{\mathcal{H}}_c \triangleq \mathfrak{C}_{(M_{\mathcal{I}}, S+n-1)}^{f,c} \{\mathbf{H}\}. \quad (11)$$

Under noiseless case, we have the following relationship:

$$\mathbf{Y}_{\mathfrak{S}}^Y (\mathbf{Y}_{\mathfrak{S}}^Y)^T = \bar{\mathcal{H}}_c \mathbf{Y}_{\mathfrak{S}}^X (\mathbf{Y}_{\mathfrak{S}}^X)^T \bar{\mathcal{H}}_c^T \quad (12)$$

In order to obtain the uniqueness results, we also need the following assumption.

Assumption 1 The matrix $\mathbf{Y}_{\mathfrak{S}}^X (\mathbf{Y}_{\mathfrak{S}}^X)^T$ has full rank.

Assumption 1 also implies that the unknown projections has sufficient angular disparity. The more irregular the specimen are, the easier to meet the assumption. Now, let us consider an eigen-decomposition of $\mathbf{Y}_{\mathfrak{S}}^Y (\mathbf{Y}_{\mathfrak{S}}^Y)^T$. Let ν denotes the cardinality of its null-space and $\mathbf{g}^{(k)} \in \mathbb{R}^{M_{\mathcal{O}}S \times 1}, k = 1, \dots, \nu$ denotes the corresponding eigenvectors. Since the column space of $\bar{\mathcal{H}}_c$ spans $\mathbf{Y}_{\mathfrak{S}}$, we can easily show the following orthogonality relationship:

$$\left(\mathbf{g}^{(k)} \right)^T \bar{\mathcal{H}}_c = \mathbf{0} \Leftrightarrow \left(\mathbf{g}^{(k)} \right)^T \bar{\mathcal{H}}_c \bar{\mathcal{H}}_c^T \mathbf{g}^{(k)} = 0. \quad (13)$$

for all $k = 1, \dots, \nu$. Hence, the resulting optimization problem for estimating 2-D PSF \mathbf{H} in (r, z) -plane is given as:

$$\text{argmin}_{\mathbf{H}} \sum_{k=1}^{\nu} \left(\mathbf{g}^{(k)} \right)^T \bar{\mathcal{H}}_c \bar{\mathcal{H}}_c^T \mathbf{g}^{(k)}. \quad (14)$$

Furthermore, $\mathbf{g}^{(k)} = \left[\left(\mathbf{g}_1^{(k)} \right)^T, \dots, \left(\mathbf{g}_S^{(k)} \right)^T \right]^T$, where $\mathbf{g}_j^{(k)} \in \mathbb{R}^{M_{\mathcal{O}} \times 1}$ denotes a subvector of $\mathbf{g}^{(k)}$. Therefore, Eq. (14) can be simplified using the following commutative property of Toeplitz structure in $\mathbf{C}_{M_{\mathcal{I}}}\{\mathbf{h}_j\}^T$:

$$\mathbf{C}_{M_{\mathcal{I}}}\{\mathbf{h}_j\}^T \mathbf{g}_i^{(k)} = \left(\mathbf{G}_i^{(k)} \right)^T \mathbf{h}_j, \quad (15)$$

where $\mathbf{G}_i^{(k)}$ is given by

$$\begin{bmatrix} \mathbf{g}_i^{(k)}(1) & \mathbf{g}_i^{(k)}(2) & \dots & \mathbf{g}_i^{(k)}(M_{\mathcal{I}}) \\ \mathbf{g}_i^{(k)}(2) & \mathbf{g}_i^{(k)}(3) & \dots & \mathbf{g}_i^{(k)}(M_{\mathcal{I}} + 1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_i^{(k)}(m) & \mathbf{g}_i^{(k)}(m + 1) & \dots & \mathbf{g}_i^{(k)}(M_{\mathcal{I}} + m - 1) \end{bmatrix}.$$

Using Eq. (15), we have

$$\bar{\mathcal{H}}_c^T \mathbf{g}^{(k)} = \left(\mathcal{G}^{(k)} \right)^T \text{VEC}\{\mathbf{H}\} \quad (16)$$

where $\mathcal{G}^{(k)}$ is defined as

$$\begin{bmatrix} \mathbf{0} & \dots & \mathbf{G}_1^{(k)} & \dots & \mathbf{G}_S^{(k)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{G}_1^{(k)} & \dots & \mathbf{G}_n^{(k)} & \dots & \mathbf{G}_S^{(k)} & \mathbf{0} \end{bmatrix}. \quad (17)$$

Therefore, the final optimization problem is reduced to

$$\text{argmin}_{\|\text{VEC}\{\mathbf{H}\}\|_2=1} \text{VEC}\{\mathbf{H}\}^T \left(\sum_{k=1}^{\nu} \mathcal{G}^{(k)} \left(\mathcal{G}^{(k)} \right)^T \right) \text{VEC}\{\mathbf{H}\}. \quad (18)$$

The solution of (18) is an eivenvector corresponding a minimum eigenvalue of $\sum_{k=1}^{\nu} \mathcal{G}^{(k)} \left(\mathcal{G}^{(k)} \right)^T$. Hence, we finally has the following identifiability condition for filter in partial data problem:

Proposition 1 (Identifiability of Filter) Let $S \geq \frac{M_T - m + 1}{m - 1} (n - 1) + 1$. Suppose, furthermore, Assumption 1 is satisfied and a generic filter \mathbf{H} has unit norm. Then, the unknown generic filter \mathbf{H} can be uniquely estimated using Eq. (18).

5. IMPLEMENTATION

We assume that the support size of PSF is known *a priori*. However, this should be estimated in practice. There exist several methods for filter size estimation such as eigenvalue based techniques [4] and residual based techniques [2]. A detailed discussion of filter size estimation is beyond the scope of this paper, and in this study, we simply chose the filter size by trial and error.

In contrary to our previous algorithm [1] that employs the Radon domain restoration, this paper applies a spatial domain restoration to reduce computation burden of inverse Radon transform especially for large images such as in real experiments. To do this, the 3-D PSF in Cartesian coordinate is interpolated from the estimated 2-D PSF in (r, z) -plane by exploiting the cylindrical symmetry. Finally, we use an iterative 3-D non-blind estimation method that maximizes the likelihood of the resulting image under noise, which is provided by MATLAB image processing toolbox (MathWorks, Natick).

6. RESULT

We construct 3-D cylindrical symmetric PSF whose (r, z) -plane image is as illustrated in Fig. 1(a). The ground-truth axial and lateral section images of 3-D phantom are illustrated in the leftmost column of Fig. 2, and the degraded image are illustrated in the middle column of Fig. 2, respectively. The dimension of 3-D phantom was $(255 \times 255 \times 41)$. Since the FOV does not cover the whole (x, y) -plane, it corresponds to the partial data case. The estimated PSF in (r, z) -plane illustrated in Fig. 1(b) was identical with the simulated PSF even though there was information loss in (x, y) -plane.

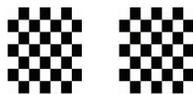


Fig. 1. PSFs in (r, z) -plane under the partial data condition. (a) Ground-truth PSF, and (b) estimated PSF.

In this simulation, the 3-D PSF was interpolated from 2-D PSF and restoration results were obtained with a non-blind deconvolution method. Due to the effect of interpolation, the restoration result are slightly degraded, but still significantly improved from the degraded images as illustrated in the rightmost column of Fig. 2.

7. CONCLUSION

This paper introduced an exact single channel 3-D blind deconvolution algorithm for rotationally symmetric point spread

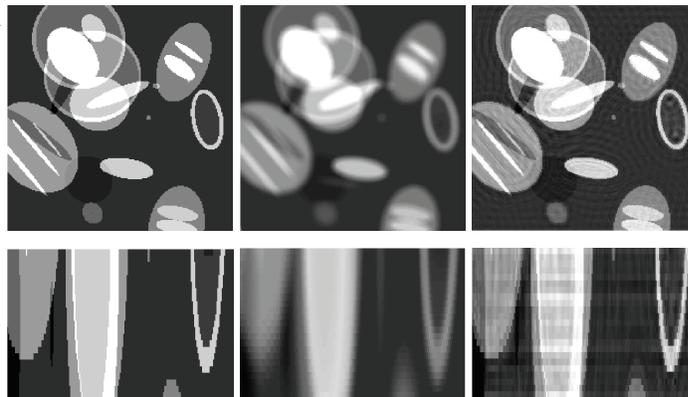


Fig. 2. Lateral and axial section images for partial data case. Left column: ground-truth, center column: blurred images, right column: restored image, respectively.

functions, which is common in optical microscopy with circular aperture. By exploiting the block Toeplitz structure of 2-D convolution matrix, we showed that the exact estimation of cylindrical symmetric point spread function is possible as long as there exists enough working distance in microscopy to cover the volumetric samples from top to bottom. Experimental results demonstrated the effectiveness of our algorithms.

8. REFERENCES

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