NON-ITERATIVE EXACT INVERSE SCATTERING USING SIMULTANEOUS ORTHOGONAL MATCHING PURSUIT (S-OMP)

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ABSTRACT

Even though recently proposed time-reversal MUSIC approach for inverse scattering problem is non-iterative and exact, the approach breaks down when there are more targets than sensors. The main contribution of this paper is a novel non-iterative exact inverse scattering algorithm that still guarantees the exact recovery of the extended targets under a very relaxed constraint on the number of source and receivers, where the conventional time-reversal MUSIC fails. Such breakthrough was possible from the observation that the induced currents on the unknown targets assume the same sparse support, which can be recovered accurately using the simultaneous orthogonal matching pursuit developed for multiple measurement vector problems. Simulation results demonstrate that perfect reconstruction can be quickly obtained from a very limited number of samples.

Index Terms— inverse scattering, time-reversal MUSIC, compressed sensing, simultaneous OMP, exact reconstruction

1. INTRODUCTION

One of the fundamental questions in wave scattering theory is to retrieve unknown constitutive properties from measured scattering fields. Such problem - often called inverse scattering - has numerous applications in practise, such as radar imaging, geophysical applications, ultrasound, and etc [1].

In particular, we are concerned about inverse scattering in the framework of the inhomoegenous Helmholtz equation:

$$\left(\nabla^2 + k^2(\mathbf{r})\right)\psi^{(n)}(\mathbf{r}) = S^{(n)}(\mathbf{r}), \quad n = 1, \cdots, N_t \;, \quad (1)$$

where $\psi^{(n)}(\mathbf{r})$ is the scalar wave field measured at detector location $\mathbf{r} \in \Gamma_r \subset \mathbb{R}^3$ produced by the n-th transmitter configurations $S^{(n)}(\mathbf{r})$, and the inhomegeneous wavenumber $k^2(\mathbf{r})$ is given by $k^2(\mathbf{r}) = k_0^2 - X(\mathbf{r})$, where k_0^2 is the known wavenumber of the background medium and $X(\mathbf{r})$ is the extended scattering potential:

$$X(\mathbf{r}) = \begin{cases} V(\mathbf{r}), & \text{if } \mathbf{r} \in D. \\ 0, & \text{otherwise;} \end{cases} \quad (2)$$

Then, Eq. (1) can be converted into the well-known Foldy-Lax multiple scattering model:

$$\psi^{(n)}(\mathbf{r}) \;=\; \psi_i^{(n)}(\mathbf{r}) + \int_\Omega d\mathbf{r}' G_0(\mathbf{r};\mathbf{r}')\psi^{(n)}(\mathbf{r})X(\mathbf{r}'), (3)$$

where $G_0(\mathbf{r};\mathbf{r}')$ denotes the homogenous Green's function calculated by

$$\left(\nabla^2 + k_0^2\right) G_0(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \;, \quad (4)$$

and the homogenous incident field $\psi_i^{(n)}(\mathbf{r})$ due to the n-th transmitter pattern is given by

$$\left(\nabla^2 + k_0^2\right)\psi_i^{(n)}(\mathbf{r}) = S^{(n)}(\mathbf{r}) \;, \quad n = 1, \cdots, N_t. \quad (5)$$

The inverse scattering problem is then to estimate *unknown* scattering potential $V(\mathbf{r})$ and its *unknown* domain D from the scattered field measurements $\psi_{scat}^{(n)}(\mathbf{r}) = \psi^{(n)}(\mathbf{r}) - \psi_i^{(n)}(\mathbf{r})$ at the detectors locations at $\mathbf{r} \in \Gamma_r$.

Note that the inverse problem in Eq. (3) is highly nonlinear since the total flux $\psi^{(n)}(\mathbf{r}')$ within the integral is a function of the unknown scattering potential $X(\mathbf{r})$. Most of the classical inverse scattering methods take the form of iterative method that uses the successive Born approximation for each newly estimated scattering potential. However, such approach is computationally expensive due to the repeated calculation of the wave equations.

In order to deal with these issues, various methods have been proposed. Especially important technique is the time-reversal approach using multiple signal classification (MUSIC) [2]. Even though this approach is non-iterative and exact within the context of multiple scattering, the time-reversal MUSIC breaks down when there are more targets than sensors. Specifically, the performance of the time-reversal MUSIC is governed by the *minimum* number of transmitters and detectors rather than their maximum; hence, the algorithm

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cannot take advantage of the increasing number of transmitters (or receivers) as long as the number of its counterpart is fixed. This *asymmetric* imaging geometry is quite often encountered in practise, such as passive radar imaging, near field optical microscopy, and etc.

The main contribution of this paper is to overcome the drawbacks of the conventional time-reversal MUSIC using the joint sparse model with common sparsity support in distributed compressed sensing [3], or multiple measurement vector (MMV) problems [4]. Unlike the conventional time-reversal MUSIC approaches, our sparsity based reconstruction theory guarantees that the maximum number of recoverable targets is limited by the *average* of the number of source and detectors, which is usually much greater than its minimum. Furthermore, the problem can be solved using various simultaneous sparse approximation algorithms such as simultaneous orthogonal matching pursuit (S-OMP) [4], and the unknown scattering potential can be exactly calculated based on the estimated support.

2. PROBLEM FORMULATION

In our model, the unknown field and scattering potential are represented as a linear combination of the basis functions $\{\alpha_i(\mathbf{r})\}_{i=1}^N$:

$$\psi^{(n)}(\mathbf{r}) = \sum_{j=1}^N \psi_j^{(n)}\alpha_j(\mathbf{r}) \quad , \quad X(\mathbf{r}) = \sum_{k=1}^N x_k^{(n)}\alpha_k(\mathbf{r}), \tag{6}$$

Also define the generalized sampling function $\{s_i(\mathbf{r})\}_{i=1}^{N_r}$:

$$\psi_{i,scat}^{(n)} = \left\langle s_i, \psi_{scat}^{(N)} \right\rangle = \int_\Omega s_i(\mathbf{r})\psi_{scat}^{(n)}(\mathbf{r})d\mathbf{r}, \quad i = 1, \cdots, N_r \,. \tag{7}$$

Then, multiple scattering model can be represented by

$$\psi_{i,scat}^{(n)} \quad = \quad \sum_{j=1}^N G_{i,j}J_j^{(n)} \tag{8}$$

where $J_j^{(n)} = \psi_j^{(n)}x_j$ and

$$G_{i,j} = \int_\Omega d\mathbf{r} \int_\Omega d\mathbf{r}' s_i(\mathbf{r})G_0(\mathbf{r};\mathbf{r}')\alpha_j(\mathbf{r})\alpha_j(\mathbf{r}') \,. \tag{9}$$

We now collect all the measurements from each transmitter pattern and receivers into a matrix form, resulting in a matrix equation:

$$\mathbf{\Psi} = \mathbf{G}\mathbf{J} \tag{10}$$

where the scattering measurement matrix is given by

$$\mathbf{\Psi} \quad = \quad \begin{pmatrix} \psi_{1,scat}^{(1)} & \psi_{1,scat}^{(2)} & \cdots & \psi_{1,scat}^{(N_t)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N_r,scat}^{(1)} & \psi_{N_r,scat}^{(2)} & \cdots & \psi_{N_r,scat}^{(N_t)} \end{pmatrix} \tag{11}$$

and the induced current matrix $\mathbf{J}$ can be constructed as follows:

$$\mathbf{J} \quad = \quad \begin{pmatrix} J_1^{(1)} & J_1^{(2)} & \cdots & J_1^{(N_t)} \\ \vdots & \vdots & \ddots & \vdots \\ J_N^{(1)} & J_N^{(2)} & \cdots & J_N^{(N_t)} \end{pmatrix} \in \mathbb{C}^{N\times N_t} \tag{12}$$

Now let us define the following *row-diversity* measure that counts the number of rows in $\mathbf{J}$ that contains non-zero elements:

$$\mathcal{R}(\mathbf{J}) = \sum_{i=1}^N \chi\left[||\mathbf{J}_{i,:}|| > 0\right] \tag{13}$$

where $\chi[\cdot]$ denotes the indicator function and $||\cdot||$ is an arbitrary vector norm. We now define an *active index set* $\mathcal{I}$ of the extended target:

$$\mathcal{I} = \{j \in \{1, 2, \cdots, N\} : x_j > 0\} \ , \tag{14}$$

and M denotes the cardinality of the active index set, i.e. $M = |\mathcal{I}|$. Now, our nonlinear inverse scattering problem becomes the following simultaneous estimation problem of the active index and the induced currents:

$$\{\hat{\mathbf{J}}_\mathcal{I}, \hat{\mathcal{I}}\} \quad = \quad \arg\min_{\mathbf{J},\mathcal{I}} ||\mathbf{\Psi} - \mathbf{G}_{:,\mathcal{I}}\mathbf{J}_\mathcal{I}||_F^2 \tag{15}$$

where $\mathbf{G}_{:,\mathcal{I}}$ denotes the submatrix of $\mathbf{G}$ by collecting only the columns corresponding the index set $\mathcal{I}$, and $\mathbf{J}_\mathcal{I}$ is the corresponding active sub-vector, and $||\cdot||_F$ denotes the Frobenius norm, respectively. The formulation Eq. (15) is the basis of the time-reversal MUSIC [2].

3. TWO STEP EXACT RECONSTRUCTION

3.1. Identifiability of Active Index Set

In many situations, the support of the targets are usually sparse compared to the whole field of view Ω. In this case, the estimation problem can be reformulated as follows:

$$(\mathbf{P0}): \quad \min \mathcal{R}(\mathbf{J}), \quad \text{subject to } \mathbf{\Psi} = \mathbf{G}\mathbf{J}. \tag{16}$$

where $\mathbf{\Psi} \in \mathbb{C}^{N_r\times N_t}$, $\mathbf{G} \in \mathbb{C}^{N_r\times N}$, $\mathbf{J} \in \mathbb{C}^{N\times N_t}$ and $\mathcal{R}(\mathbf{J})$ is the row diversity. Given matrix $\mathbf{G}$, we now define the quantity $spark(\mathbf{G})$ as the smallest number of linearly dependent column of $\mathbf{G}$. Then, Chen and Huo [4] showed the following sufficient condition for the uniqueness of **(P0)**.

Theorem 1 *Let $rank(\mathbf{\Psi})$ denotes the rank of the matrix $\mathbf{\Psi}$. Then,* **(P0)** *has the unique solution if*

$$\mathcal{R}(\mathbf{J}) \quad \leq \quad \left(spark(\mathbf{G}) + rank(\mathbf{\Psi}) - 1\right)/2 \,. \tag{17}$$

Theorem 1 gives us the insight about ultimate number of targets that the inverse scattering approach **(P0)** can reconstruct. Specifically, using $rank(\mathbf{\Psi}) \leq N_t$ and $spark(\mathbf{G}) \leq N_r + 1$, we can easily show that the maximum number of targets that **(P0)** has unique solution is given by:

$$\mathcal{R}(\mathbf{J}) \quad \leq \quad \left(N_t + N_r\right)/2 \,. \tag{18}$$

Note that this is a very big improvement over the time-reversal MUSIC since the maximum number of recoverable targets in time-reversal MUSIC is given by $\min(N_t, N_r)$ [2]. For the case of asymmetric imaging example with $N_t \ll N_r$, we can see that $(N_t + N_r)/2 \gg \min(N_t, N_r)$; hence, the maximum number of recoverable targets in **(P0)** is much bigger than that of the time-reversal MUSIC.

The upper bound Eq. (17) for the recoverable number of target can be also used for optimizing the imaging geometry. We first define another important concept called *mutual incoherence*:

$$\mu(\mathbf{G}) = \max_{1 \leq i,j \leq N, i \neq j} |\langle \mathbf{G}_{:,i}, \mathbf{G}_{:,j} \rangle| \ , \tag{19}$$

where $\mathbf{G}_{:,i}$ denotes the i-th column of the $\mathbf{G}$ matrix. Then, Chen and Huo [4] showed that following:

Corollary 1 *If* $\mathbf{\Psi} = \mathbf{GJ}$ *and*

$$\mathcal{R}(\mathbf{J}) \quad < \quad \left(\mu(\mathbf{G})^{-1} + rank(\mathbf{\Psi})\right)/2 \tag{20}$$

then matrix $\mathbf{J}$ *is the unique solution to the problem* **(P0)**.

Now, in order to maximize the number of recoverable targets, we should maximize both $\mu(\mathbf{G})^{-1}$ and $rank(\mathbf{\Psi})$. One of the nice properties of this criterion is that the geometry of the sources and the detectors are completely decoupled. More specifically, we first optimize the detector locations by minimizing the mutual incoherence $\mu(\mathbf{G})$. This procedure is completely independent not only from the source geometry but also from the unknown target geometry. After the detector geometry is fixed, the remaining optimization criterion is to maximize $rank(\mathbf{\Psi})$ by changing the source configuration.

3.2. Sparse Reconstruction under Simultaneous Sparsity

Even though **(P0)** gives us the useful information about the uniqueness of the sparse reconstruction, its direct optimization requires computationally expensive combinatorial optimization. For the single measurement vector case such that $N_t = 1$, there have been extensive investigations about so-called l_0/l_1-*equivalence* [5]. More specifically, if the measurement basis and the signal basis are *incoherent*, the convex l_1 optimization problem provides the unique solution which is identical to that of **(P0)** [5].

The main technical difficulty to extend this idea to multiple measurement vector cases (i.e. $N_t > 2$) is that the corresponding l_1 norm of $N \times N_t$ matrix $\mathbf{J}$ does not exist. Recently, many different approaches has been proposed to address this issue [4, 6]. The most popular approaches are based on forward sequential selection methodsusing simultaneous orthogonal matching pursuit (S-OMP) [4].

Algorithm S-OMP

1. Initialize the residual matrix $\mathbf{R}_0 = \mathbf{\Psi}$, the index set $\mathcal{I}_0 = \varnothing$, and the iteration count $t = 1$.
2. Find the index ω_t that solves the optimization problem:
$$\max_{\omega} || \left(\mathbf{G}_{:,\omega}\right)^H \mathbf{R}_t||_p \tag{21}$$
where $p \geq 1$ and $|| \cdot ||_p$ denotes the l_p norm.
3. Set $\mathcal{I}_t = \mathcal{I}_{t-1} \cup \{\omega_t\}$.
4. Determine the orthogonal projector $\mathbf{P}_t$ onto the span of the atoms indexed in $\mathcal{I}_t$:
$$\mathbf{P}_t = \mathbf{G}_{:,\mathcal{I}_t} \left(\mathbf{G}^H_{:,\mathcal{I}_t} \mathbf{G}_{:,\mathcal{I}_t}\right)^{-1} \mathbf{G}^H_{:,\mathcal{I}_t} \tag{22}$$
5. Calculate the new residual:
$$\mathbf{R}_t \quad = \quad (\mathbf{I} - \mathbf{P}_t)\, \mathbf{\Psi} \tag{23}$$
6. Increase the step count t and go to step 2 if $||\mathbf{R}_t||_F \geq \epsilon$.

Note that except for the l_p-norm of the vector in Eq. (21), the algorithm is identical to the standard OMP algorithm for the single measurement vector. Chen and Huo [4] showed that regardless of the vector norm the S-OMP can successfully recover the sparsest representation.

3.3. Exact Reconstruction of Scattering Potential

For the estimated active index set $\hat{\mathcal{I}}$, the unknown induced current matrix $\mathbf{J}$ can be calculated using least square fitting:

$$\hat{\mathbf{J}}_{\hat{\mathcal{I}}} = \left(\mathbf{G}_{:,\hat{\mathcal{I}}}\right)^{\dagger} \mathbf{\Psi} \tag{24}$$

In continuous formulation, this implies that the both unknown induced current $J^{(n)}(\mathbf{r}') = \psi^{(n)}(\mathbf{r}')V(\mathbf{r}'), \mathbf{r}' \in D$ and the unknown domain D due to the n-th transmitter pattern have been estimated. Then, the unknown total flux $\psi^{(n)}(\mathbf{r}''), \mathbf{r}'' \in \Omega$ can be estimated using the following Foldy-Lax multiple scattering model:

$$\hat{\psi}^{(n)}(\mathbf{r}'') \quad = \quad \psi^{(n)}_{inc}(\mathbf{r}'') + \int_{\hat{D}} d\mathbf{r}' G_0(\mathbf{r}'', \mathbf{r}') \hat{J}^{(n)}(\mathbf{r}') \tag{25}$$

Using the estimated total field $\hat{\psi}^{(n)}(\mathbf{r}'')$, the least squares estimate for the unknown scattering potential is given by

$$\hat{V}(\mathbf{r}'') = \frac{\sum_{n=1}^{N_t} \left(\hat{\psi}^{(n)}(\mathbf{r}'')\right)^* \hat{J}^{(n)}(\mathbf{r}'')}{\sum_{n=1}^{N_t} |\hat{\psi}^{(n)}(\mathbf{r}'')|^2}, \quad \mathbf{r}'' \in D. \tag{26}$$

Marengo et al [2] derived different form of non-iterative exact recontruction algorithm for the scattering potential when the support D is given. However, the algorithm in [2] is for the point targets and we are not aware of generalization for the extended targets. Our prefect reconstruction formula Eq. (26) is so general that it can be used for both point targets as well as extended targets.

4. NUMERICAL RESULTS

The first simulation was performed to verify the effects of the increasing number of transmitters. The dimension of the FOV is set to $4\lambda \times 4\lambda \times 4\lambda$ with the voxel size of $0.2\lambda \times 0.2\lambda \times 0.2\lambda$, where λ denotes the wavelength. On each surrounding planes, 100 detectors are uniformly distributed with the detector pitch of 0.4λ. Figure 1 illustrates that more targets can be reconstructed perfectly with increasing number of transmitters, which confirms our theoretical analysis. Furthermore, this number of recoverable target is significantly larger than that of the time-reversal MUSIC, since the maximum number of recoverable targets by the time-reversal MUSIC is only $N_t(= 1, 2, \cdots, 6)$ for this imaging scenario.

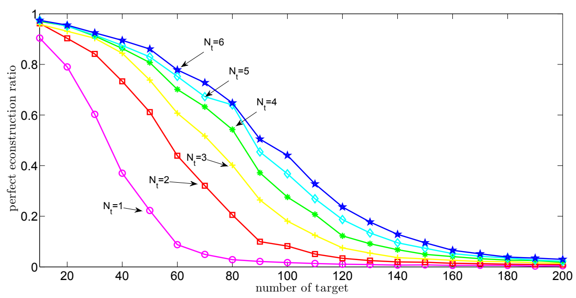

Fig. 1. Perfect reconstruction ratio of the proposed algorithm.

We have also performed the simulation for more realistic imaging scenario, where the imaging object (say, "K"-object) is an extended target as illustrated in Fig. 2(a). The FOV is $10\lambda \times 10\lambda \times 10\lambda$ with the voxel size of $0.5\lambda \times 0.5\lambda \times 0.5\lambda$. Again, 100 detectors are distributed uniformly on the surrounding planes, and the transmitters are located at the center of the surrounding planes. Figures 2(b)-(d) are the reconstructed image using one, two and six transmitters, respectively. We can observe that more accurate reconstruction can be obtained with more transmitters. Note that in this imaging geometry $N_t \leq 6$ and $N_r = 600$; hence, the maximum number of recoverable target is smaller than $(600 + 6)/2 = 303$. Currently, the S-OMP approach can recover about the size of non-zero support of the "K"-object, which is about 50. The gap between the theoretical performance bound and S-OMP performance may be due to the large mutual coherence especially below the diffraction limit. Hence, to achieve the maximum performance even below the sub-diffraction resolution, we may need to devise additional incoherent measurement scheme.

5. CONCLUSION

This paper derived a novel non-iterative exact inverse scattering algorithm using simultaneous OMP based on the observation that the induced current on the unknown targets assumes the same sparse support. Using the theoretical and numerical analysis, we showed that the maximum number of recoverable targets using the new method is upper bounded by the average of the number of transmitters and detectors. This is a significantly improvement over the conventional time-reversal MUSIC especially when the number of source or detectors are significantly smaller than its counterpart. We believe that our results fill the missing gap toward the complete theory for the non-iterative exact inverse scattering theory.

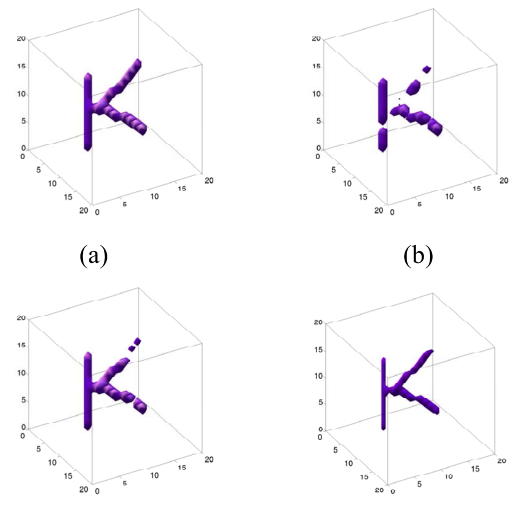

Fig. 2. (a) Original phantom. S-OMP reconstruction using (b) one, (c) two , and (d) six transmitters, respectively.

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