JOINT SPARSITY RECOVERY METHOD FOR THE EIT PROBLEM TO RECONSTRUCT ANOMALIES

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ABSTRACT

This paper considers an electrical impedance tomography (EIT) problem to reconstruct multiple small anomalies from boundary measurements. The inverse problem of EIT is a severely ill-posed nonlinear inverse problem so that the conventional methods usually require linear approximation or iterative procedure. In this paper, we propose a non-iterative reconstruction method by exploiting the joint sparsity to attack these problems. It consists of three steps; first, the target location and corresponding current values are reconstructed using the joint sparse recovery. Second, the unknown potential is estimated, and conductivities are calculated as a final step. The advantages of the proposed method over conventional approaches are accuracy and speed, and we validate these effectiveness of the proposed algorithm by numerical simulations.

Index Terms— Electrical impedance tomography, joint sparsity, small anomalies, non-iterative recovery

1. INTRODUCTION

Electrical impedance tomography (EIT) is a noninvasive imaging technique that reconstructs the electrical property of the medium based on the voltage measurements around the boundary of the medium from the injected currents. With the help of relatively low cost imaging equipment, EIT has been applied for various clinical applications such as monitoring lung problems or heart function and detecting for the breast cancer [1]. However, the inverse problem of EIT is nonlinear and ill-posed due to the nonlinear coupling of the unknown potential to the electrical property and compactness of the mapping.

To circumvent the nonlinearity, linear approximation or an iterative method is commonly used [1]. Linear approximation employs the internal potential values for the background electrical property instead of the unknown potential values, so the reconstruction time is quite fast; however, it produces an approximation error. On the other hand, the iterative method solves the forward problem multiple times to update the electrical property and the unknown potential for more accurate reconstruction result; but, it suffers from ill-posedness and the computational burden. To resolve the ill-posedness, various algorithms have been proposed for reconstruction of anomalies, such as the topological derivative algorithm proposed by Eschenauer et al [2] for shape optimization. Even though various reconstruction methods have been developed, most of them are, however, within the framework of either linear approximation or iterative methods.

In this paper, we propose a non-iterative reconstruction method for EIT problem based on the joint sparsity [3]. It exploits the sparsely distributed anomalies and multiple current injections to change the original non-linear inverse problem to joint sparse recovery problem. The idea originally comes from the previous researches that solve the nonlinearity in the inverse problem of diffuse optical tomography [4, 5], and now we apply it to EIT problem. More specifically, we will show that the reconstruction problem can be solved using the following three steps. First, the non-zero position and corresponding induced current values are reconstructed from the joint sparse recovery. Second, the unknown potential is estimated. Finally, the electrical properties are calculated as a linear problem. Simulation results validate the efficiency of the proposed method and show more accurate and faster recovery compared to the linear approximation without approximation and the iterative updating procedures.

2. THEORY

2.1. Problem Formulation

Let Ω be a bounded domain in $\mathbb{R}^d$, say $d = 2$ for two dimensional problem[1]. Suppose that Ω contains a finite number of small anomalies, say $D_1, \ldots, D_N$ (N is the number of anomalies). We suppose that the conductivity of the background is 1, and that of D_j is σ_j ($j = 1, \ldots, N$) which is assumed to be unknown. So the conductivity profile of the

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[1]Even though we only deal with two dimensions, it is straightforward to extend it to three dimensions.

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domain Ω is given by

$$\sigma = \chi(\Omega \setminus \cup_{j=1}^{N}) + \sum_{j=1}^{N} \sigma_j \chi(D_j), \tag{1}$$

where $\chi(D)$ is the characteristic function of D.

The EIT problem is to find the locations, some geometric information, and the conductivities of D_j's using a finite number of pairs of voltages (Dirichlet data) and currents (Neumann data) on the boundary of Ω. More precisely, for given $g_1, \ldots, g_M \in L_0^2(\partial\Omega)$ (M is the number of distinct current injection on the boundary), let u_k be the solution to the Neumann boundary value problem

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \sigma \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g_k, \displaystyle\int_{\partial\Omega} u \, d\sigma = 0, \end{cases} \tag{2}$$

and let U_k be the solution in absence of anomalies. Then it is known that the following formula holds for $x \in \partial\Omega$[6]:

$$v_k(x) = \sum_{j=1}^{N} (1 - \sigma_j) \int_{D_j} \nabla_y \Gamma(x - y) \cdot \nabla u_k(y) dy, \quad x \in \partial\Omega. \tag{3}$$

where, $v_k(x) = \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right)[(u_k - U_k)|_{\partial\Omega}](x)$. Here, $\mathcal{K}_{\partial\Omega}$ is a Neumann-Poincaré operator given by:

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle y - x, \nu_y \rangle}{|x - y|^2} \varphi(y) \, d\sigma(y) \ , \quad x \in \partial\Omega, \tag{4}$$

and $\Gamma(x)$ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \frac{1}{2\pi} \ln|x| \ .$$

So the problem is to find location of D_j (and σ_j) using v_k for $k = 1, \ldots M$.

2.2. Proposed Method

This section describes the proposed method for EIT problem using the joint sparsity. Let us define the current on the anomalies as

$$I_k(y) = \begin{cases} (1 - \sigma_p)\nabla u_k(y) & \text{for } y \in D_p, \ p = 1, \ldots, N, \\ 0 & \text{for } y \in \Omega \setminus \cup_{p=1}^{N} D_p. \end{cases} \tag{5}$$

Then the formula (3) becomes

$$v_k(x) = \int_{\Omega} \nabla_y \Gamma(x - y) \cdot I_k(y) \, dy, \quad x \in \partial\Omega. \tag{6}$$

Note that the anomalies are located at fixed positions despite of the different current injections, whereas the current value I_k on the anomalies vary. Therefore, assuming the sparsity for the support set $\cup_{p=1}^{N} D_p$, the problem (6) becomes a joint sparsity problem. More specifically, let us assume that I_k, $1 \leq k \leq M$ can be expressed with either piecewise constant functions or by splines:

$$I_k(y) = \begin{bmatrix} \displaystyle\sum_{j=1}^{n} I_{k,1}\left(y_{(j)}\right) b_1\left(y, y_{(j)}\right) \\ \displaystyle\sum_{j=1}^{n} I_{k,2}\left(y_{(j)}\right) b_2\left(y, y_{(j)}\right) \end{bmatrix}, \quad y \in \Omega, \tag{7}$$

where b_d, $d = 1, 2$, is the basis function for the d-th coordinate and $\{y_{(j)}\}_{j=1}^{n}$ is the discretized points of Ω. After substituting (7) into (6), we have the following matrix equation:

$$Y = AX + E = [A_1, A_2] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + E, \tag{8}$$

with the measurement noise $E \in \mathbb{R}^{m \times M}$. We define, respectively, the sensing matrix $A = [A_1, A_2] \in \mathbb{R}^{m \times 2n}$, the currents $X = [X_1^T, X_2^T]^T \in \mathbb{R}^{2n \times M}$ and the measurements $Y \in \mathbb{R}^{m \times M}$ as

$$(A_d)_{ij} = \tilde{\Gamma}_d(x_{(i)}, y_{(j)}), \quad (X_d)_{jk} = I_{k,d}(y_{(j)}), \quad \text{and } (Y)_{ik} = v_k(x_{(i)}),$$

where $\{x_{(i)}\}_{i=1}^{m}$ is the collection of the measurement points on $\partial\Omega$ and

$$\tilde{\Gamma}_d(x, y_{(j)}) = \int_{\Omega} \nabla_{y,d} \Gamma(x - y) b_d\left(y, y_{(j)}\right) dy.$$

Here, $\nabla_{y,d}\Gamma(x - y)$ ($d = 1, 2$) is the d-th coordinate component of $\nabla_y \Gamma(x-y)$. The solution X to (8) has a pairwise joint sparsity implying that X_1 and X_2 are nonzero at the same rows which correspond to the positions where the anomalies are located. Based on this equation, we can formulate the following joint sparse recovery problem [3]:

$$\min_{X} \ ||X||_0, \ \text{ subject to } \ ||Y - AX||_F^2 \leq \epsilon, \tag{9}$$

where $||X||_0$ denotes the number of rows that have non-zero elements in X and $||\cdot||_F$ denotes the Frobenius norm.

We can use any of the joint sparse recovery algorithms to solve (9) and estimate the positions of the anomalies, $\hat{D} = \{\hat{D}_1, \ldots, \hat{D}_{\hat{N}}\}$, and the values of $\hat{I}_{k,d}(y_{(j)})$ for $y_{(j)} \in \hat{D}$. After that, the unknown solution u_k can be easily estimated using the following recursive relationship:

$$\hat{u}_k(x) = U_k(x) + \mathcal{D}_{\Omega}[(u_k - U_k)|_{\partial\Omega}](x) - \int_{\hat{D}} \nabla_y \Gamma(x-y) \cdot \hat{I}_k(y) \ dy, \quad x \in \hat{D}. \tag{10}$$

Using the estimated values, the conductivity σ_p's can be calculated from the following equation:

$$v_k(x) = \int_{\hat{D}} (1 - \sigma_p) \nabla_y \Gamma(x-y) \cdot \nabla \hat{u}_k(y) \ dy, \quad k = 1, \ldots, M, \ x \in \partial\Omega. \tag{11}$$

We emphasize that (11) is now linear. The unknown value of u_k is estimated in (10) so that neither linear approximation nor the iterative update is required. Furthermore, we can expect more efficient and less ill-posed reconstruction procedure due to the knowledge of the estimated position of anomalies.

3. IMPLEMENTATIONS

3.1. Joint Sparse Recovery

To solve the problem (9), we use the multiple sparse Bayesian learning (M-SBL) algorithm [7]. The M-SBL algorithm assumes that X follows the *i.i.d.* Normal distribution with $\text{vec}(X) \sim \mathcal{N}(0, I_M \otimes \Gamma)$, where I_M denotes the $M \times M$ identity matrix and Γ is the diagonal matrix whose i-th element is γ_i, the common variance component for the i-th row values of X. Considering the pairwise joint sparsity in (9), we further assume $\gamma_i = \gamma_{i+n}$ for $i = 1, 2, \cdots, n$. As a result of the M-SBL algorithm, we obtain the solution X to (9). We now calculate a spectrum of the current as follows:

$$p_{(j)} = \sqrt{\sum_{d=1}^{2}\sum_{k=1}^{M}\left[(X_d)_{jk}\right]^2}, \quad \text{for } 1 \le j \le n, \tag{12}$$

and then estimate the position of the anomalies $\hat{D}$ by thresholding the normalized spectrum, $p_{(j)}/\max(p)$. One can apply a preconditioning procedure before the M-SBL algorithm to deal with the ill-posedness in the inversion of the sensing matrix.

3.2. Conductivity Recovery

After solving the joint sparse recovery problem in (9) and finding the unknown value of $u_k(x)$ using (10), we can calculate the conductivity of the anomalies from the linear equation in (11). Let us denote by $\{\hat{y}_{(j)}\}_{j=1}^{\tilde{n}}$ the estimated points of $\hat{D}$ and δ the area of the discretized grid, then the discretized version of (11) is as follows:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_M \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_{\tilde{n}} \end{bmatrix} = Ax, \tag{13}$$

where $y_k \in \mathbb{R}^{m\times 1}$, $A_k \in \mathbb{R}^{m\times \tilde{n}}$, and $x \in \mathbb{R}^{\tilde{n}\times 1}$ are given by

$$(y_k)_i = v_k(x_{(i)}), \quad (A_k)_{i,j} = -\nabla_y\Gamma(x_{(i)} - \hat{y}_{(j)}) \cdot \nabla\hat{u}_k(\hat{y}_{(j)})\,\delta,$$

and $x_j = \sigma(\hat{y}_{(j)}) - 1$, respectively. To solve (13), we use the following constrained optimization problem with l^1 penalty for noise robust reconstruction:

$$\begin{aligned} &\arg\min_x && ||x||_1 \\ &\text{subject to} && ||Ax - y||_2 \le \epsilon. \end{aligned} \tag{14}$$

This method can be also used for conventional Born approximation, which uses the background solution U_k, instead of u_k. In other words, the sensing matrix A in (13) becomes $A_k \in \mathbb{R}^{m\times n}$, $1 \le k \le M$, such that

$$(A_k)_{ij} = -\nabla_y\Gamma(x_{(i)} - y_{(j)}) \cdot \nabla U_k(y_{(j)})\,\delta \quad \text{with } y_{(j)} \in \Omega.$$

In this case, the size of sensing matrix is increased back from $\tilde{n}$ to n due to the lack of information on the position of anomalies, and their values are inaccurate due to the approximation error. Note that the advantage of the Born approximation over the conventional iterative methods is the computation speed. However, the proposed method has even higher computational efficiency compared to the Born approximation. We show the explicit reconstruction time and accuracy for various simulation results in the next section. To solve the problem (14), we exploit a constrained split augmented Lagrangian shrinkage algorithm (C-SALSA) [8].

3.3. Extended Target Recovery

The proposed method is designed for the recovery of sparse anomalies, however, the second and third steps are unrelated to the size of the target. Therefore, the proposed method can be also applied to the extended target as long as one can solve the first step. To alleviate this challenging problem for the extended target, we suggest to solve the following equation to estimate the currents based on the estimated $\hat{D}$ from the M-SBL algorithm:

$$v_k(x) = \int_{\hat{D}} \nabla_y\Gamma(x - y) \cdot I_k(y)\,dy, \quad x \in \partial\Omega. \tag{15}$$

4. RESULTS

In this section we present numerical simulation results of the proposed method and conventional linear approximation (the Born approximation). We show two examples of sparse anomalies (Case A) and an extended anomaly (Case B) as illustrated in Fig.1 and consider it as the problem of finding tumors and their conductivities in the biological medium. In all cases, Ω is an ellipse of semi-major and semi-minor axes 10 and 7, and the background conductivity is 1. The left anomaly in the first case has the conductivity value of 2 and the right one has the value of 5. In the second case, the conductivity of the anomaly is 5. The field of view is dicretized to have a grid size 0.5 for reconstruction. In all cases, it is

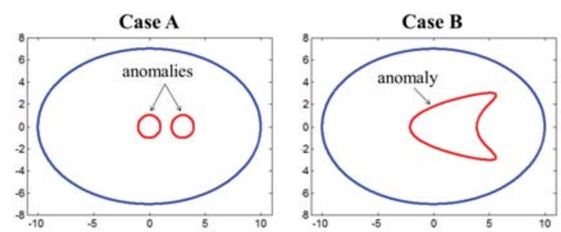

Fig. 1. Simulation geometry.

used only two measurement vectors, the Dirichlet data of the solution to u_k, for the two given currents g_1 and g_2 on $\partial\Omega$ given by

$$g_1 = (1, 0) \cdot \nu_\Omega \quad \text{and} \quad g_2 = (0, 1) \cdot \nu_\Omega.$$

Here ν_Ω is the outward unit normal vector to $\partial\Omega$. In order to acquire the measurement data, we use the computation method modified from that in [9], which is based on the expression of u_k and U_k in terms of the single layer potentials. See [9] for the details of the numerical code. For all cases, the number of nodal points on each $\partial\Omega$ and ∂D_j is 2000 in the direct solver, and we use the data of 100 uniformly sampled measurement points in the reconstruction procedure. For the quantitative measure, we calculate the relative error defined as

$$\text{error} = \frac{||x_{true} - x_{recon}||^2}{||x_{true}||^2}, \tag{16}$$

where $x_{true} = \sigma_{true} - 1$, $x_{recon} = \sigma_{recon} - 1$. Gaussian noise with a SNR of $40dB$ was added to the measurement points. We normalized the sensing matrix A in (9) and (13) for each column to have a unit l^2 norm before applying the algorithms, and the results were obtained by averaging 20 trials with independent noise realizations.

The average reconstructed image from the 20 trials for two cases are illustrated in Fig.2. We set the ground truth in Fig.2(a) as the difference of the conductivity in anomalies and the background. The anomalies are clearly separated and have conductivity values close to the ground truth for the results using the proposed method in Fig.2(c) compared to those of the Born approximation in Fig.2(b). The relative error is calculated based on (16) for two cases and summarized in Table.1. The value in the table denotes the average relative error from 20 trials and the value in the parenthesis is their standard deviation. As we can see, the proposed method shows lower error compared to the Born approximation for both cases.

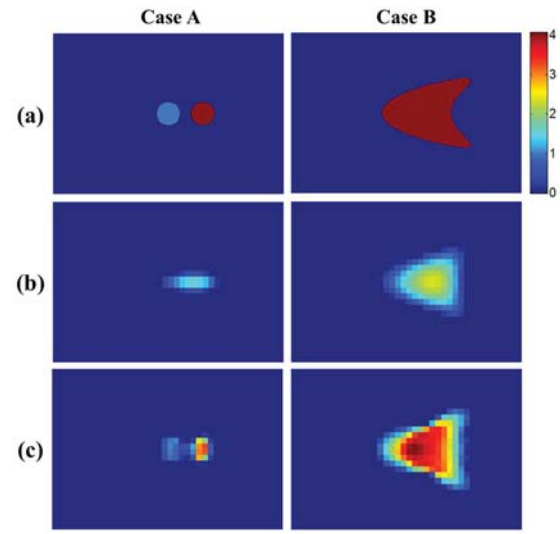

Fig. 2. Average reconstructed image ($\sigma - 1$) for 20 trials of two cases. (a) Ground truth, (b) Born approximation, and (c) proposed method.

	Case A	Case B
Proposed Method	0.3697 ($\pm$ 0.1166)	0.2227 ($\pm$ 0.0123)
Born Approximation	0.6368 ($\pm$ 0.0087)	0.5096 ($\pm$ 0.0046)

Table 1. Relative error for the reconstruction results.

The average reconstruction time (in [sec]) is summarized in Table.2 (using a PC with CPU : core i7 sandy bridge). As we described, the run time of calculating the conductivity of the anomalies (C-SALSA algorithm) using the proposed method is faster than that of the Born approximation even though it has an additional step of finding non-zero support using M-SBL algorithm. The difference between total run time and C-SALSA (and M-SBL) is mostly dedicated for generating the sensing matrix.

	Proposed method			Born Approximation	
	M-SBL	C-SALSA	Total	C-SALSA	Total
Case A	0.145	0.103	0.705	0.830	1.290
Case B	0.135	0.117	0.724	1.349	1.801

Table 2. Average run time [sec] of the reconstruction.

5. CONCLUSION

This paper proposed the non-iterative reconstruction method in EIT problem using the joint sparsity. We showed that the non-linear inverse problem of EIT can be changed to the joint sparse recovery problem and it enables the estimation of the unknown internal potential values. After that, the conductivity of the anomalies can be calculated from the proposed linear problem. The simulation results showed that the proposed method outperforms over the conventional Born approximation.

6. REFERENCES

[1] M. Cheney, D. Isaacson, and J. C. Newell, "Electrical impedance tomography," *SIAM Rev.*, vol. 41, no. 1, pp. 85–101, 1999.

[2] H. A. Eschenauer, V. V. Kobelev, and A. Schumacher, "Bubble method for topology and shape optimization of structures," *Struct. Optimization*, vol. 8, pp. 42–51, 1994.

[3] J. Chen and X. Huo, "Theoretical results on sparse representations of multiple measurement vectors," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4634–4643, 2006.

[4] O. K. Lee, J. M. Kim, Y. Bresler, and J. C. Ye, "Compressive diffuse optical tomography: non-iterative exact reconstruction using joint sparsity," *IEEE Trans. Med. Imag.*, vol. 30, no. 5, pp. 1129–1142, 2011.

[5] O. Lee and J. C. Ye, "Joint sparsity-driven non-iterative simultaneous reconstruction of absorption and scattering in diffuse optical tomography," *Opt. Express*, vol. 21, pp. 26589–26604, 2013.

[6] H. Ammari and H. Kang, *Polarization and moment tensors with applications to inverse problems and effective medium theory*, Springer-Verlag, New York, 2007, Applied Mathematical Sciences, Vol. 162.

[7] D. P. Wipf and B. D. Rao, "An empirical Bayesian strategy for solving the simultaneous sparse approximation problem," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3704–3716, 2007.

[8] M. Afonso, J. Bioucas-Dias, and M. Figueiredo, "An augmented Lagrangian approach to the constrained optimization formulation of imaging inverse problems," *IEEE Trans. Image Process.*, vol. 20, no. 3, pp. 681–695, March 2011.

[9] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Solna, and H. Wang, *Mathematical and statistical methods for multistatic imaging*, Springer, 2013, Lecture Notes in Math. 2098.