COMPRESSIVE SUBSPACE FITTING FOR MULTIPLE MEASUREMENT VECTORS

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ABSTRACT

We study a multiple measurement vector problem (MMV), where multiple signals share a common sparse support and are sampled by a common sensing matrix. While a diversity gain from joint sparsity had been demonstrated earlier in the case of a convex relaxation method using a mixed norm, only recently was it shown that similar gain can be achieved by greedy algorithms if we combine greedy steps with a MUSIC-like subspace criterion. However, the main limitation of these hybrid algorithms is that they require a large number of snapshots or a high signal-to-noise ratio (SNR) for an accurate subspace as well as partial support estimation. Hence, in this work, we show that the noise robustness of these algorithms can be significantly improved by allowing sequential subspace estimation and support filtering, even when the number of snapshots is insufficient. Numerical simulations show that the proposed algorithms significantly outperform the existing greedy algorithms and are quite comparable with computationally expensive state-of-art algorithms.

Index Terms— Compressed sensing, multiple measurement vector problems, subspace estimation, greedy algorithm

1. INTRODUCTION

We study a multiple measurement vector (MMV) problem, where multiple signals share a common sparse support set and each signal is measured by a common measurement matrix. An MMV problem is one way in which multiple correlated signals can appear in a signal ensemble, and also has many important applications [1]. A central theme in these studies has been that joint sparsity within signal ensembles enables a reduction in the number of required measurements [2].

Recently, Kim *et al.* [3] and Lee *et al.* [4] independently showed that such a diversity gain can be also achieved in a new class of greedy algorithms by exploiting the so-called generalized multiple signal classification (MUSIC) criterion. More specifically, these algorithms obtain a partial support estimate using a conventional MMV greedy algorithm, and then the atoms corresponding to the partial supports are augmented into a data matrix to obtain an augmented signal subspace estimate. Finally, a MUSIC-like criterion is derived for the augmented subspace to find the remaining support. The performance improvement of these greedy algorithms is substantial and nearly achieves the l_0 bound when a signal subspace and partial support estimation are accurate due to a sufficient number of snapshots or high signal to noise ratio (SNR) [3, 4]. However, if either of these estimation is erroneous owing to an insufficient number of snapshots or low SNR, performance degrades. Similar observations have been made in the literature on classical array signal processing [5].

While increasing the number of snapshots is relatively easier in classical sensor array signal processing problems, in some MMV problems such as parallel MR imaging [6], an additional snapshot requires a hardware change by adding a new receiver coil. Hence, to exploite other dimensions would be beneficial. We are aware that joint sparse recovery methods such as Bayesian approaches [7, 8], or convex optimization techniques [9], are shown to be statistically robust in the direction of arrival estimation problems[8]. However, these approaches are usually computationally expensive for MMV problems with a large number of sensors, so we need a new greedy algorithm that achieves an optimal performance with a significantly reduced computational complexity.

Therefore, one of the main goals of this paper is to address how these hybrid greedy methods can be made robust without increasing the number of snapshots. One important contribution is a new theory explaining that the generalized MUSIC criterion is a special case of a new subspace criterion that can be used to derive two sequential strategies to improve the accuracy of an augmented signal subspace estimation. More specifically, a forward greedy subspace estimation step improves the robustness of an augmented signal subspace estimation by adding newly discovered atoms in the MUSIC step, whereas the backward support filtering provides additional robustness by eliminating the inaccurate portion of support estimates. By combining the two steps, we develop a novel sequential CS-MUSIC algorithm that is robust, even with a limited number of snapshots. Using numerical simulations, we show that the sequential CS-MUSIC is superior to the existing subspace-based greedy algorithms and exhibits similar performance behavior to the mixed-norm [10] or Bayesian approaches [7] with a significantly reduced computational complexity.

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2. GENERALIZED SUBSPACE CRITERION

Throughout the paper, $\mathbf{x}^i$ and $\mathbf{x}_j$ correspond to the i-th row and j-th column of matrix X. When I is an index set, X^I, A_I corresponds to a submatrix collecting corresponding rows of X and columns of A, respectively. The rows (or columns) in $\mathbb{R}^n$ are in *general position* if any n collection of rows (or columns) are linearly independent.

Definition 1 (Canonical form noiseless MMV [3]) *Suppose we are given a sensing matrix $A \in \mathbb{R}^{m\times n}$ and an observation matrix $B \in \mathbb{R}^{m\times r}$ such that $B = AX_*$ for some $X_* \in \mathbb{R}^{n\times r}$ and $\|X_*\| = |\mathrm{supp}X| = k$, where m, n, and r are positive integers ($r \leq m < n$) that represent the number of sensor elements, an ambient space dimension, and the rank of an observation matrix, respectively. A canonical form noiseless MMV problem is given as an estimation problem of k-sparse vectors $X \in \mathbb{R}^{n\times r}$ using the following formula:*

$$\text{minimize} \quad \|X\|_0, \quad \text{subject to} \quad B = AX, \tag{1}$$

where $\|X\|_0 = |\mathrm{supp}X|$, $\mathrm{supp}X = \{1 \leq i \leq n : \mathbf{x}^i \neq 0\}$, and the observation matrix B is full rank.

In this section, we assume that an MMV problem is of the canonical form. However, this assumption will be relaxed later in noise analysis. Moreover, throughout the paper, we assume that the nonzero rows of X are in general position.

Note that the generalized MUSIC criterion in [3] requires $\delta^L_{2k-r+1}(A) < 1$. This implies that, if a sensing matrix is obtained from a random Gaussian and if measurement is noiseless, then we have the following minimal sampling rate [3]:

$$m \geq (1+\delta)(2k-r+1) \text{ for some } \delta > 0.$$

If we have a redundant sampling $m \gg 2k-r+1$, the following theorem can be used instead as the extension of the generalized MUSIC criterion in [3].

Theorem 1 *Suppose $1 \leq l \leq r$ and we have a canonical MMV model $AX = B$ with a sensing matrix A that satisfies an RIP condition with $\delta^L_{2k-r+l}(A) < 1$. Then, for a given index set I such that $|I| \leq \min(2(k-r)+l, k)$ and $|I \setminus \mathrm{supp}X| \leq k-r+l$, then*

$$|I \cap \mathrm{supp}X| \geq k-r+1 \iff \mathrm{rank}[A_I\ B] < |I| + r. \tag{2}$$

Note that the conditions for the index set I in Theorem 1 says that multiple index sets I can exist for a given l. Furthermore, if we choose $|I| = k-r+1$, Theorem 1 is reduced to the following generalized MUSIC criterion in [3].

Corollary 1 *[3] Suppose we have a canonical MMV model $AX = B$ with a sensing matrix A that satisfies an RIP condition with $\delta^L_{2k-r+1}(A) < 1$. Then, for $I_{k-r} \subset \mathrm{supp}X$ with $|I_{k-r}| = k-r$ and any $j \in \{1, \cdots, n\} \setminus I_{k-r}$, we have*

$$j \in \mathrm{supp}X \iff \mathbf{a}_j^* P^{\perp}_{R([A_{I_{k-r}}\ B])}\mathbf{a}_j = 0.$$

In addition, the main advantage of the generalized subspace fitting criterion is its flexibility to be extended for more general situation.

3. SEQUENTIAL CS-MUSIC ALGORITHM

3.1. Forward Greedy: Sequential Subspace Estimation

In [3], the CS-MUSIC first determines $k-r$ indices of $\mathrm{supp}X$ with CS-based algorithms such as 2-thresholding or S-OMP, and then it recovers the remaining r indices of $\mathrm{supp}X$ using the generalized MUSIC criterion. For this, a projection operator onto the noise subspace is calculated as the orthogonal complement of the augmented signal subspace $R([A_{I_{k-r}}\ B])$. However, the following result can further extend the existing generalized MUSIC criterion [3] .

Theorem 2 *Suppose $1 \leq l \leq r$ and we have a canonical MMV model $AX = B$ with a sensing matrix A that satisfies an RIP condition with $\delta^L_{2k-r+l}(A) < 1$. Then, if we have an index set I such that $|I| \leq \min(2(k-r)+l-1, k-1)$, $|I \setminus \mathrm{supp}X| \leq k-r+l-1$ and $|I \cap \mathrm{supp}X| \geq k-r$,*

$$j \in \mathrm{supp}X \iff \mathbf{a}_j^* P^{\perp}_{R([A_I\ B])}\mathbf{a}_j = 0 \text{ for } j \notin I.$$

Note that $R([A_I\ B]) = R([A_{I_{k-r}}\ B])$ and $\dim R([A_{I_{k-r}}\ B]) = k$ for all $I \subset \mathrm{supp}X$ and $k-r \leq |I|$. This implies that we first need to find I_{k-r} support using a compressive sensing algorithm, then we augment newly added supports into the initial estimate I_{k-r}. As is shown in [11], such a greedy procedure improves the accuracy of the augmented signal subspace estimation. Theorem 2 leads us to the following sequential algorithm (SeqSubspace), as in Table I. Note that the algorithm can be combined with any joint sparse recovery algorithm that provides a $k-r$ initial support estimate.

Table 1.

Algorithm: I= SeqSubspace(A, B, I_{k-r})
- Set $q = 0$ and $I = I_{k-r}$.
- While $q < r$, do the following procedure:
1. Perform an SVD of $[A_I\ B] = [U_1, U_0]\mathrm{diag}[\Sigma_1, \Sigma_0][V_1, V_0]^*$, where $\Sigma_1 = \mathrm{diag}[\sigma_1, \cdots, \sigma_k]$, $\Sigma_0 = \mathrm{diag}[\sigma_{k+1}, \cdots, \sigma_{k+q}]$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{k+q}$.
2. Take $j_q = \arg\min_{j\notin I} \|P^{\perp}_{R(U_1)}\mathbf{a}_j\|^2$.
3. Set $I := I \cup \{j_q\}$, let $q := q+1$ and goto step 1.
- Return I.

3.2. Backward Greedy: Support Filtering

As discussed before, we can easily expect that the performance of the generalized MUSIC step is highly dependent on the selection of $k-r$ correct indices of the support of X. In practice, even though the first consecutive steps may not provide all true partial supports, it is more likely that among a k-sparse support estimate of S-OMP, part of the supports (not in sequential order) can be correct. Hence, if the estimate of the support of X has at least $k-r$ indices of the support of X and we can identify them, then we can expect that the performance of the compressive MUSIC will be improved.

Theorem 3 (Backward support filtering criterion) *Suppose $1 \le l \le r$ and we have a canonical MMV model $AX = B$ with a sensing matrix A that satisfies an RIP condition with $0 \le \delta^L_{2k-r+l}(A) < 1$. Then, if we have an index set I such that $|I| \le \min(2(k-r)+l, k)$, $|I \setminus \mathrm{supp}X| \le k-r+l$ and $|I \cap \mathrm{supp}X| \ge k-r+1$, then we have for $j \in I$,*

$$j \in \mathrm{supp}X \iff \mathbf{a}_j^* P^{\perp}_{R([A_{I\setminus\{j\}}\ B])}\mathbf{a}_j = 0.$$

Theorem 3 requires that $k-r+1$ supports (not in sequential order) out of a larger support estimate is correct. Then, the location of a correct $k-r$ support can be readily estimated using the following backward support filtering. Compared to a forward greedy procedure that improves the accuracy of the signal subspace estimation, the backward support filtering can improve the accuracy of a partial support recovery, and the accuracy of an augmented signal subspace.

Table 2.

Algorithm: I_{k-r}= $\mathsf{SupportFiltering}(A, B, I)$
- For all $j \in I$, calculate the quantities $\zeta(j) := \lVert P^{\perp}_{R([A_{I\setminus\{j\}}\ B])}\mathbf{a}_j \rVert^2$. - Making an ascending ordering of $\zeta(j)$ for $j \in I$, choose indices that correspond to the first $k-r$ indices and put these indices into I_{k-r}. - Return I_{k-r}.

By combining the forward and the backward greedy steps, this paper develops the following sequential CS-MUSIC algorithm decribed in Table III.

Table 3.

Algorithm: I_k= $\mathsf{SeqCSMUSIC}(A, Y, k, r)$ Input: k, r, $A \in \Re^{m\times n}$, $Y \in \Re^{m\times N}$ Output: k support estimate I_k
- Estimate the k support estimate I_k of supp$X$ using *any* MMV algorithm. - U := Rank-r signal subspace estimate of $R(Y)$. - I_{k-r} :=$\mathsf{SupportFiltering}(A, U, I_k)$. - I_k :=$\mathsf{SeqSubspace}(A, U, I_{k-r})$. - Return I_k.

4. NUMERICAL RESULTS

4.1. Dependency on Snapshot Number

First, we demonstrate that a sequential CS-MUSIC is less sensitive to the number of snapshots. The simulation parameters were as follows: $m \in \{1, 2, \ldots, 30\}$, $n = 128, k = 8, r = 4$, and $N \in \{6, 16, 256\}$, respectively. The elements of a sensing matrix A were generated from a Gaussian distribution $\mathcal{N}(0, 1/m)$, and then each column of A was normalized to have an unit norm. An unknown signal X with $\mathrm{rank}(X) = r$ was generated using the same procedure as in [4]. Specifically, we randomly generated a support I, and then the corresponding nonzero signal components were obtained by

$$X^I = \Psi\Phi\ , \tag{3}$$

where $\Psi \in \mathbb{R}^{k\times r}$ and Λ were set to random orthonormal columns, and $\Phi \in \mathbb{R}^{r\times N}$ were made using Gaussian distribution $\mathcal{N}(0, 1/N)$. After generating noiseless data, we added zero mean white Gaussian noise to have $\mathsf{SNR} = 30dB$. We declared success if an estimated support was the same as $\mathrm{supp}X$, and success rates were averaged over 1000 experiments. Fig. 1 shows success rates of a sequential CS-MUSIC compared to that of CS-MUSIC or SA-MUSIC. As shown in Fig 1, sequential CS-MUSIC exhibits nearly similar recovery performance for various snapshot numbers, whereas the original form of CS-MUSIC/SA-MUSIC requires a large number of snapshots to achieve maximum performance.

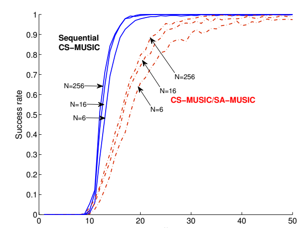

Fig. 1. Snapshot dependent performance behaviour of the sequential CS-MUSIC and the original CS-MUSIC/SA-MUSIC. The simulation parameters are $n = 128, k = 8, r = 4$ and $SNR = 30dB$.

4.2. Performance Comparison with State-of-Art Joint Sparse Recovery Algorithms

To compare the proposed algorithm with various state-of-art joint sparse recovery methods, the recovery rates of various state-of-art joint sparse recovery algorithms such as CS-MUSIC/SA-MUSIC, l_1/l_2 mixed norm approaches [12], and M-SBL [7], are plotted in Fig. 2 along with those of a sequential CS-MUSIC. Since the mixed norm approach and M-SBL do not provide a exact k-sparse solution, we used the support for the largest k coefficients as a support estimate in calculating the perfect recovery ratio. Figs. 2(a) and (b) show the recovery rates for $N = 8$ and 256, respectively. Sequential CS-MUSIC outperforms S-OMP and the original CS-MUSIC/SA-MUSIC consistently, and its performance nearly achieves those of M-SBL and the mixed norm approaches. Note that the performance of M-SBL and the mixed norm approaches were identical. Indeed, the additional sampling cost for a sequential CS-MUSIC compared to the M-SBL or the mixed norm approaches is very small. Considering that any subspace method needs additional redundancy (i.e. $m \geq k + 1$) to avoid ambiguity in the signal subspace estimation, we believe that sequential CS-MUSIC nearly achieves the optimum performance. Furthermore, this high performance can be achieved at negligible computational complexity. Note that the complexity of the sequential CS-MUSIC is only a fraction of those of M-SBL and the mixed norm approaches, as shown in Figs. 2(c)(d) for $N = 8$ and 256, respectively.

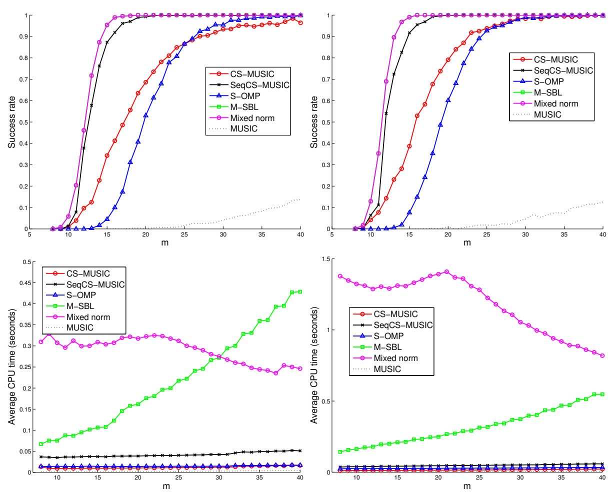

Fig. 2. Performance of various joint sparse recovery algorithm at (a) $N = 8$ and (b) $N = 256$, when $n = 128, k = 8, r = 4$, and $SNR = 30dB$. (c)(d) Average CPU time for $N = 8$ and $N = 256$, respectively.

5. CONCLUSION

In this paper, we derived two greedy strategies to improve the noise robustness of recent hybrid joint sparse recovery algorithms such as CS-MUSIC and SA-MUSIC. Furthermore, we demonstrated that even with limited number of snapshots, there are two different ways to improve the noise robustness of augmented signal subspace estimation: one by sequential subspace estimation and the other by filtering out incorrect support. We further explained that the two greedy steps are byproducts of a novel generalized subspace criterion. Numerical simulation demonstrated that the new algorithm consistently outperforms the existing greedy algorithms and nearly achieves optimal performance with minimal computational complexity.

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